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Asymptotically linear system of three equations near resonance

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Abstract

This paper deals with the asymptotically linear system

$$\left.\begin{array}{ll} -\Delta u_1 = \lambda\theta_1 u_3 + f_1(\lambda, x, u_1, u_2, u_3) & \text{in } \Omega \\ -\Delta u_2 = \lambda\theta_2 u_2 + f_2(\lambda, x, u_1, u_2, u_3) & \text{in } \Omega \\ -\Delta u_3 = \lambda\theta_3 u_1 + f_3(\lambda, x, u_1, u_2, u_3) & \text{in } \Omega \\ \quad u_1 = u_2 = u_3 = 0 \quad \text{on } \partial\Omega\,, & \end{array}\right\}$$

where $\theta_i > 0$ for $i = 1, 2, 3$ with $\theta_2 \neq \sqrt{\theta_1\theta_3}$, λ is a real parameter and $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. The linear part of the system has two simple eigenvalues with nonnegative eigenfunctions each with at least one zero component. We provide sufficient conditions which guarantee bifurcation from infinity of positive solutions from both, one or none of the two simple eigenvalues. Under additional assumptions on the nonlinear perturbations, we determine the λ-direction of bifurcation as well. We use bifurcation theory to establish our results.

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1. Introduction

The purpose of this article is to use bifurcation theory to study positive solutions of elliptic system of the form

$$\left.\begin{array}{rl} -\Delta u_1 = \lambda\theta_1 u_3 + f_1(\lambda, x, u_1, u_2, u_3) & \text{in } \Omega \\ -\Delta u_2 = \lambda\theta_2 u_2 + f_2(\lambda, x, u_1, u_2, u_3) & \text{in } \Omega \\ -\Delta u_3 = \lambda\theta_3 u_1 + f_3(\lambda, x, u_1, u_2, u_3) & \text{in } \Omega \\ u_1 = u_2 = u_3 = 0 \quad \text{on } \partial\Omega\,, & \end{array}\right\} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N > 1)$ is a bounded domain with $C^{2,\xi}$-boundary $\partial\Omega$ for some $\xi \in (0, 1)$ and a bounded open interval if $N = 1$. Throughout the paper, we will assume that $\lambda \in \mathbb{R}$ is a parameter and $\theta_i > 0$, $i = 1, 2, 3$ are constants such that

$$\theta_2 \neq \sqrt{\theta_1\theta_3}. \tag{1.2}$$

For $i = 1, 2, 3$, the nonlinear perturbations $f_i : \mathbb{R} \times \Omega \times \mathbb{R}^3 \to \mathbb{R}$ satisfy the following assumptions uniformly on compact intervals of λ:

(H1) f_i are Carathéodory functions, and
(H2) there exists $q \in L^r(\Omega)$, $r > N$, with $q(x) \geq 0$ for a.e. $x \in \Omega$ such that

$$|f_i(\lambda, x, s_1, s_2, s_3)| \leq q(x) \quad \text{for all } (s_1, s_2, s_3) \in \mathbb{R}^3.$$

Due to condition (H2), our system (1.1) is asymptotically linear at infinity. Such problems have been studied using bifurcation theory. For example, see [3–5,20,21] for results in scalar case, and [6] for system of two equations. To the best of our knowledge, this paper is the first to deal with asymptotically linear systems of three equations using bifurcation theory. See [1,6,12,13,15–17, 23–25] where asymptotically linear systems of two equations were studied by other methods.

It is well known that for asymptotically linear problems, bifurcation from infinity can occur only at an eigenvalue of the linear part (see [14]). If it occurs at an eigenvalue λ^*, then the solutions near the bifurcation point are smooth perturbations of eigenfunctions corresponding to λ^* (see Proposition 4.1). Thus if bifurcation (from infinity) of positive solutions occurs, then it must occur only at eigenvalues with nonnegative eigenfunctions.

Under some appropriate conditions on θ_i $(i = 1, 2, 3)$, we show that the linear part of (1.1) has three simple eigenvalues but only two of these have eigenfunctions that are componentwise nonnegative (see Proposition 2.1). In this article, we discuss some *new* and *unexpected phenomena* which occur when these eigenfunctions have at least one *zero* component. Presence of the zero component in both eigenfunctions gives rise to the following possibilities for (1.1):

(a) *Bifurcation from infinity of positive solution at both of these simple eigenvalues.*
(b) *Bifurcation from infinity of positive solution at only one of these two simple eigenvalues.*
(c) *No bifurcation from infinity of positive solution at any* $\lambda \in \mathbb{R}$.