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On the nonstationary Stokes system in a cone

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Abstract

The authors consider the Dirichlet problem for the nonstationary Stokes system in a threedimensional cone. They obtain existence and uniqueness results for solutions in weighted Sobolev spaces and prove a regularity assertion for the solutions.

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0. Introduction

The present paper deals with the Dirichlet problem for the nonstationary Stokes system in a threedimensional cone K. This means, we consider the problem

$$\frac{\partial u}{\partial t} - \Delta u + \nabla p = f, \quad -\nabla \cdot u = g \quad \text{in } K \times (0,\infty), \tag{1}$$

$$u(x,t) = 0 \quad \text{for } x \in \partial K,\ t > 0, \quad u(x,0) = 0 \quad \text{for } x \in K. \tag{2}$$

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The goal of the paper is to prove the existence and uniqueness of solutions in weighted Sobolev spaces and a regularity assertion for the solutions. A theory for the heat equation in domains with conical points and edges was established in a number of papers in the last 30 years, see [2,7, 10–12,16,18,19]. This theory involves in particular existence and uniqueness results for solutions in weighted Sobolev and Hölder spaces, regularity assertions and the asymptotics of the solutions near vertices and edges. A class of general parabolic problems in a cone was studied in [4–6]. However, this class of problems does not include the Stokes system. Although the stationary Stokes system in domains with conical points and edges is well-studied (see, e.g., [3,14,15]), there is still no theory for the nonstationary Stokes system in domains with singular boundary points. The present paper is a first step in developing such a theory.

An essential part of the paper (Sections 1 and 2) consists of the investigation of the parameter-depending problem

$$s\,\tilde{u} - \Delta\tilde{u} + \nabla\tilde{p} = \tilde{f}, \quad -\nabla\cdot\tilde{u} = \tilde{g} \quad \text{in } K, \quad \tilde{u} = 0 \quad \text{on } \partial K, \tag{3}$$

which arises after the Laplace transformation with respect to the time t. In Section 1, we prove that this problem has a uniquely determined variational solution $(\tilde{u}, \tilde{p}) \in E^1_\beta(K) \times \big(V^0_\beta(K) + V^1_\beta(K)\big)$ if $\operatorname{Re} s \ge 0$, $s \ne 0$ and $|\beta|$ is sufficiently small. Here $V^l_\beta(K)$ denotes the weighted Sobolev space of all functions (vector-functions) with finite norm

$$\|u\|_{V^l_\beta(K)} = \Big(\int\limits_K \sum_{|\alpha|\le l} r^{2(\beta-l+|\alpha|)} \big|\partial_x^\alpha u(x)\big|^2\, dx \Big)^{1/2}, \tag{4}$$

while $E^l_\beta(K)$ is the weighted Sobolev space with the norm

$$\|u\|_{E^l_\beta(K)} = \Big(\int\limits_K \sum_{|\alpha|\le l} \big(r^{2\beta} + r^{2(\beta-l+|\alpha|)}\big) \big|\partial_x^\alpha u(x)\big|^2\, dx \Big)^{1/2}, \tag{5}$$

$r = r(x)$ denotes the distance of the point x from the vertex of the cone.

The goal of Section 2 is to prove the existence and uniqueness of solutions $(\tilde{u}, \tilde{p}) \in E^2_\beta(K) \times V^1_\beta(K)$ of the parameter-depending problem in the case $\operatorname{Re} s \ge 0$, $s \ne 0$. Note that the problem (3) is <u>not</u> elliptic with parameter in the sense of [1]. Therefore, the results concerning general parabolic problems in a cone which were obtained in [4–6] are not applicable to our problem.

We get two β-intervals for which we have an existence and uniqueness result in the space $E^2_\beta(K) \times V^1_\beta(K)$, namely the intervals

$$\frac{1}{2} - \lambda_1^+ < \beta < \frac{1}{2} \quad \text{and} \quad \frac{1}{2} < \beta < \min\Big(\mu_2^+ + \frac{1}{2}, \lambda_1^+ + \frac{3}{2}\Big) \tag{6}$$

(see Theorems 2.2 and 2.3). Here, λ_1^+ and μ_2^+ are positive numbers depending on the cone. More precisely, λ_1^+ is the smallest positive eigenvalues of the operator pencil $\mathcal{L}(\lambda)$ generated by the Dirichlet problem for the stationary Stokes system, while μ_2^+ is the smallest positive eigenvalue of the operator pencil $\mathcal{N}(\lambda)$ generated by the Neumann problem for the Laplacian, respectively. We show that the above inequalities for β are sharp.