Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 260 (2016) 8388–8402

Journal of Differential Equations

www.elsevier.com/locate/jde

Two elliptic closed geodesics on positively curved Finsler spheres

Huagui Duan [1]

School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, People's Republic of China

Received 2 February 2015; revised 16 December 2015

Available online 27 February 2016

Abstract

In this paper, we prove that for every Finsler n-dimensional sphere (S^n, F) with reversibility λ and flag curvature K satisfying $\left(\frac{\lambda}{1+\lambda}\right)^2 < K \leq 1$, either there exist infinitely many closed geodesics, or there exist at least two elliptic closed geodesics and each linearized Poincaré map has at least one eigenvalue of the form $e^{\sqrt{-1}\theta}$ with θ being an irrational multiple of π.

MSC: 53C22; 58E05; 58E10

Keywords: Positively curved; Closed geodesic; Elliptic; Finsler metric; Spheres

1. Introduction and main result

A closed curve on a Finsler manifold is a closed geodesic if it is locally the shortest path connecting any two nearby points on this curve. As usual, on any Finsler manifold (M, F), a closed geodesic $c : S^1 = \mathbf{R}/\mathbf{Z} \to M$ is *prime* if it is not a multiple covering (i.e., iteration) of any other closed geodesics. Here the m-th iteration c^m of c is defined by $c^m(t) = c(mt)$. The inverse curve c^{-1} of c is defined by $c^{-1}(t) = c(1-t)$ for $t \in \mathbf{R}$. Note that unlike Riemannian

E-mail address: duanhg@nankai.edu.cn.

[1] Partially supported by NNSF (Nos. 11131004, 11471169), LPMC of MOE of China and Nankai University.

http://dx.doi.org/10.1016/j.jde.2016.02.025

manifold, the inverse curve c^{-1} of a closed geodesic c on a irreversible Finsler manifold need not be a geodesic. We call two prime closed geodesics c and d *distinct* if there is no $\theta \in (0, 1)$ such that $c(t) = d(t + \theta)$ for all $t \in \mathbf{R}$. On a reversible Finsler (or Riemannian) manifold, two closed geodesics c and d are called *geometrically distinct* if $c(S^1) \neq d(S^1)$, i.e., their image sets in M are distinct. We shall omit the word *distinct* when we talk about more than one prime closed geodesic.

For a closed geodesic c on n-dimensional manifold (M, F), denote by P_c the linearized Poincaré map of c. Then $P_c \in \mathrm{Sp}(2n-2)$ is symplectic. For any $M \in \mathrm{Sp}(2k)$, we define the *elliptic height* $e(M)$ of M to be the total algebraic multiplicity of all eigenvalues of M on the unit circle $\mathbf{U} = \{z \in \mathbf{C} |\, |z| = 1\}$ in the complex plane $\mathbf{C}$. Since M is symplectic, $e(M)$ is even and $0 \le e(M) \le 2k$. A closed geodesic c on the n-dimensional manifold (M, F) is called *elliptic* if $e(P_c) = 2(n-1)$, i.e., all the eigenvalues of P_c locate on $\mathbf{U}$; *hyperbolic* if $e(P_c) = 0$, i.e., all the eigenvalues of P_c locate away from $\mathbf{U}$; *non-degenerate* if 1 is not an eigenvalue of P_c. A Finsler manifold (M, F) is called *bumpy* if all the closed geodesics on it are non-degenerate.

There is a famous conjecture in Riemannian geometry which claims there exist infinitely many closed geodesics on any compact Riemannian manifold. This conjecture has been proved for many cases, but not yet for compact rank one symmetric spaces except for S^2. The results of Franks [11] in 1992 and Bangert [2] in 1993 imply this conjecture is true for any Riemannian 2-sphere. But once one move to the Finsler case, the conjecture becomes false. It was quite surprising when Katok [17] in 1973 found some irreversible Finsler metrics on spheres with only finitely many closed geodesics and all closed geodesics are non-degenerate and elliptic (cf. [35]).

Recently, index iterated theory of closed geodesics (cf. [4] and [21]) has been applied to study the closed geodesic problem on Finsler manifolds. For example, Bangert and Long in [3] show that there exist at least two closed geodesics on every (S^2, F). After that, a great number of multiplicity and stability results has appeared (cf. [6–10,22,23,25,30–34] and therein).

In [28], Rademacher has firstly introduced the reversibility $\lambda = \lambda(M, F)$ of a compact Finsler manifold defined by

$$\lambda = \max\{F(-X) \mid X \in TM,\ F(X) = 1\} \ge 1. \tag{1.1}$$

Then Rademacher in [29] present some results about multiplicity and the length of closed geodesics and about their stability properties. For example, let F be a Finsler metric on S^n with reversibility λ and flag curvature K satisfying $\left(\frac{\lambda}{1+\lambda}\right)^2 < K \le 1$, then there exist at least $n/2 - 1$ closed geodesics with length $< 2n\pi$. If $\frac{9\lambda^2}{4(1+\lambda)^2} < K \le 1$ and $\lambda < 2$, then there exists a closed geodesic of elliptic–parabolic type, i.e., its linearized Poincaré map splits into 2-dimensional rotations and a part whose eigenvalues are ± 1.

Recently, Wang in [31] proved that for every Finsler n-dimensional sphere S^n with reversibility λ and flag curvature K satisfying $\left(\frac{\lambda}{1+\lambda}\right)^2 < K \le 1$, either there exist infinitely many prime closed geodesics or there exists one elliptic closed geodesic whose linearized Poincaré map has at least one eigenvalue which is of the form $\exp(\pi i \mu)$ with an irrational μ. Furthermore, assume that this metric F is bumpy, in [32], Wang shows that there exist at least $2[\frac{n+1}{2}]$ closed geodesics on (S^n, F). Also in [32], Wang shows that for every bumpy Finsler metric F on S^n satisfying $\frac{9\lambda^2}{4(1+\lambda)^2} < K \le 1$, there exist two prime elliptic closed geodesics provided the number of closed geodesics on (S^n, F) is finite. Very recently, Hingston and Rademacher in [16] proved the existence of at least two distinct closed geodesics on (S^n, F) satisfying $\left(\frac{\lambda}{1+\lambda}\right)^2 < K \le 1$.