Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 260 (2016) 8403–8427

Journal of Differential Equations

www.elsevier.com/locate/jde

Stability of the train of N solitary waves for the two-component Camassa–Holm shallow water system

Xingxing Liu

Department of Mathematics, China University of Mining and Technology, Xuzhou, Jiangsu, 221116, China

Received 5 November 2015

Available online 7 March 2016

Abstract

Considered herein is the integrable two-component Camassa–Holm shallow water system derived in the context of shallow water theory, which admits blow-up solutions and the solitary waves interacting like solitons. Using modulation theory, and combining the almost monotonicity of a local version of energy with the argument on the stability of a single solitary wave, we prove that the train of N solitary waves, which are sufficiently decoupled, is orbitally stable in the energy space $H^1(\mathbb{R}) \times L^2(\mathbb{R})$.

MSC: 35G25; 35B30

Keywords: Two-component Camassa–Holm system; N solitary waves; Orbital stability

1. Introduction

In this paper, we are concerned with the following two-component Camassa–Holm shallow water system [4,12,28,37]

$$\begin{cases} m_t + 2u_x m + u m_x + \rho\rho_x = 0, \ m = u - u_{xx}, & t > 0, \ x \in \mathbb{R}, \\ \rho_t + (u\rho)_x = 0, & t > 0, \ x \in \mathbb{R}, \end{cases} \tag{1.1}$$

E-mail address: liuxxmaths@cumt.edu.cn.

http://dx.doi.org/10.1016/j.jde.2016.02.028

where the variables $u(t,x)$, $\rho(t,x)$ describe the horizontal velocity of the fluid and the horizontal deviation of the surface from equilibrium (or scalar density), respectively. The system (1.1) was originally introduced by Chen et al. [4] and Falqui in [21]. It is completely integrable [12,21,27] as it can be written as a compatibility condition of two linear systems (Lax pair). Compared with the other integrable multicomponent Camassa–Holm-type systems, the system (1.1) has caught a large amount of attention, after Constantin and Ivanov [12] derived it in the context of shallow water regime. It is noticed that the boundary assumptions $u \to 0$ and $\rho \to 1$ as $|x| \to \infty$, at any instant t, are required in their hydrodynamical derivation.

For $\rho \equiv 0$, the system (1.1) becomes the classical Camassa–Holm equation, which was first derived as an abstract bi-Hamiltonian partial differential equation by Fokas and Fuchssteiner [22]. Then Camassa and Holm [2] independently rediscovered it modeling shallow water waves with $u(t,x)$ representing the free surface over a flat bottom. Moreover, it was found by Dai [16] as a model for nonlinear waves in cylindrical hyperelastic rods where $u(t,x)$ stands for the radial stretch relative to a pre-stressed state. In the past two decades, the reason for the Camassa–Holm equation as the master equation in shallow water theory is that it gives a positive response to the question 'What mathematical models for shallow water waves could include both the phenomena of soliton interaction and wave breaking?', which was proposed by Whitham [39]. The appearance of breaking waves as one of the remarkable properties of the Camassa–Holm equation, however, can not be captured by the KdV and BBM equations [13]. Plenty of impressive known results on wave breaking for the Camassa–Holm equation have been obtained in [3,6,9,10,31,34,35,41]. Recently, we notice that Brandolese [1] unifies some of earlier results by a more natural blow-up condition, that is, local-in-space blow-up criterion which means the condition on the initial data is purely local in space variable.

On the other hand, it was shown that the Camassa–Holm equation has solitary waves interacting like solitons [2,3], which capture the essential features of the extreme water waves [7,8,11,38]. Hence many papers addressed another fundamental qualitative property of solutions for the Camassa–Holm equation, which is the stability of solitary wave solutions. As commented in [14], due to the fact that a small perturbation of a solitary wave can yield another one with a different speed and phase shift, we could only expect orbital stability for solitary waves. Constantin and Strauss [14] gave a very simple proof of the orbital stability of the peakons by using the conservation laws. Then they [15] applied the general approach developed by [23], to cope with the stability of the smooth solitary waves. A series of works by El Dika and Molinet [17–19] were devoted to the study of the stability of the train of N solitary waves, multipeakons and multi antipeakons–peakons, respectively. Moreover, Lenells [30] presented a variational proof of the stability of the periodic peakons.

For $\rho \neq 0$, the system (1.1) has also attracted much attention owing the fact that it has both solutions which blow up in finite time and solitary wave solutions interacting like solitons. The Cauchy problem of the system (1.1) has been studied extensively. The local well-posedness for the system (1.1) with initial data $(u_0, \rho_0)^t \in H^s \times H^{s-1}$, $s \geq 2$, by Kato's semigroup theory [29], was established in [20]. Then Gui and Liu [26] improved the well-posedness result with initial data in the Besov spaces (especially in $H^s \times H^{s-1}$, $s > \frac{3}{2}$). More interestingly, singularities of the solutions for the system (1.1) can occur only in the form of wave breaking, while blow-up solutions with a different class of certain initial profiles were shown in [12,20,24–26,40]. Moreover, the system (1.1) has also global strong solutions [12,24,25]. Here we recall the following global existence result needed in our developments.