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Abstract framework for the theory of statistical solutions

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Abstract

An abstract framework for the theory of statistical solutions is developed for general evolution equations, extending the theory initially developed for the three-dimensional incompressible Navier–Stokes equations. The motivation for this concept is to model the evolution of uncertainties on the initial conditions for systems which have global solutions that are not known to be unique. Both concepts of statistical solution in trajectory space and in phase space are given, and the corresponding results of existence of statistical solution for the associated initial value problems are proved. The wide applicability of the theory is illustrated with the very incompressible Navier–Stokes equations, a reaction–diffusion equation, and a nonlinear wave equation, all displaying the property of global existence of weak solutions without a known result of global uniqueness.

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1. Introduction

The concept of statistical solution was introduced for the study of turbulence in incompressible Newtonian fluid flows. In a turbulent flow, most relevant physical quantities (e.g. velocity, kinetic energy, energy dissipation) display a wild variation in space and time, while displaying a more orderly behavior when averaged in space or time (see e.g. [68,5,42,54,38,51]). This behavior appears, in fact, for different realizations of the flow, with somehow "universal" properties, so that one is led to consider averages with respect to an ensemble of flows, in an attempt to capture common properties of turbulent flows.

One can work with ensemble average in a formal sense, without worrying about the regularity of the solutions of the system, or one can be more strict and work with some notion of weak solution of the system for which existence results for the corresponding initial value problem are available. The statistical solution were introduced exactly with this later purpose in mind: they have been defined to model, in a rigorous way, the evolution of ensembles of weak solutions of the incompressible Navier–Stokes equations, as a foundation for a rigorous treatment of turbulent flows.

Since its inception, the theory of statistical solutions for the incompressible Navier–Stokes equations has been the basis for a growing number of rigorous results for turbulent flows (e.g. [29,22,8,30,39,32,31,60,63]). The concept of statistical solution has also been successfully adapted to a number of other models, particularly fluid flow models, but also other types of nonlinear partial differential equations (e.g. [14,4,77,44,67,19,74,58,17,16,23,47,56,24,46,61,13]).

In fact, the notion of ensemble average is relevant for any evolution system displaying a complicated dynamics, in which uncertainties in the initial condition are of crucial concern. The concept of ensemble averages is directly related to the evolution of a probability distribution of initial conditions. In a well-posed system, with a well-defined semigroup $\{S(t)\}_{t\geq 0}$, the evolution $\{\mu_t\}_{t\geq 0}$ of the probability distribution of the state of the system at each time t is just the transport, or push-forward, $\mu_t = S(t)\mu_0$, of the initial probability measure μ_0, by the semigroup (more precisely, $\mu_t(E) = \mu_0(S(t)^{-1}E)$, for any Borel subset E of the phase space). The difficulty is to extend this definition to obtain the distributions μ_t for systems in which $\{S(t)\}_{t\geq 0}$ might not be defined. This was the aim of the concept of statistical solution in the particular and fundamental case of the three-dimensional incompressible Navier–Stokes equations.

Two main definitions of statistical solutions have been introduced in the 1970s. First, Foias [27,28], in works stemmed from discussions with Prodi (see e.g. [34]), introduced the concept of statistical solution in phase space, consisting of a family of measures on the phase space of the Navier–Stokes system, parametrized by the time variable, and representing the evolution of the probability distribution of the state of the system. Then, Vishik and Fursikov [75,76] introduced the notion of a space–time statistical solution, which is that of a single measure defined on the space of trajectories of the system, hence encompassing both space and time variables at the same time.

Still in the mid to late 1970s, it is worth mentioning the work by Ladyzhenskaya and Vershik [49], presenting a different proof of existence of statistical solution (in phase space) using a representation theorem by Castaing [21], for the measurability of multivalued solution maps, to overcome the fact that the solution operator is not a continuous, single-valued map. There is also the work by Arsen'ev [2] using a measurable selection argument to construct a statistical solution and, in fact, already introducing a notion of space–time statistical solution, which at first had some restrictions on the initial measure, but that were eventually relaxed [3]. Another proof