Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 260 (2016) 8675–8715

Journal of Differential Equations

www.elsevier.com/locate/jde

Regularity theory for general stable operators ☆

Xavier Ros-Oton [a,*], Joaquim Serra [b]

[a] The University of Texas at Austin, Department of Mathematics, 2515 Speedway, Austin, TX 78751, USA
[b] Universitat Politècnica de Catalunya, Departament de Matemàtiques, Diagonal 647, 08028 Barcelona, Spain

Received 13 December 2014; revised 6 January 2016

Available online 9 March 2016

Abstract

We establish sharp regularity estimates for solutions to $Lu = f$ in $\Omega \subset \mathbb{R}^n$, L being the generator of any stable and symmetric Lévy process. Such nonlocal operators L depend on a finite measure on S^{n-1}, called the spectral measure.

First, we study the interior regularity of solutions to $Lu = f$ in B_1. We prove that if f is C^α then u belong to $C^{\alpha+2s}$ whenever $\alpha + 2s$ is not an integer. In case $f \in L^\infty$, we show that the solution u is C^{2s} when $s \neq 1/2$, and $C^{2s-\epsilon}$ for all $\epsilon > 0$ when $s = 1/2$.

Then, we study the boundary regularity of solutions to $Lu = f$ in Ω, $u = 0$ in $\mathbb{R}^n \setminus \Omega$, in $C^{1,1}$ domains Ω. We show that solutions u satisfy $u/d^s \in C^{s-\epsilon}(\overline{\Omega})$ for all $\epsilon > 0$, where d is the distance to $\partial\Omega$.

Finally, we show that our results are sharp by constructing two counterexamples.

MSC: 35B65; 60G52; 47G30

Keywords: Stable Lévy processes; Interior regularity; Boundary regularity

1. Introduction and results

The regularity of solutions to integro-differential equations has attracted much interest in the last years, both in the Probability and in the PDE community. This type of equations arise natu-

☆ The authors were supported by grant MTM2014-52402-C3-1-P.

* Corresponding author.
E-mail addresses: ros.oton@math.utexas.edu (X. Ros-Oton), joaquim.serra@upc.edu (J. Serra).

http://dx.doi.org/10.1016/j.jde.2016.02.033
0022-0396/

rally in the study of Lévy processes, which appear in many different situations, from Physics to Biology or Finance.

A very important class of Lévy processes are the α-stable processes, with $\alpha \in (0, 2)$; see [4] and [32]. These are processes satisfying self-similarity properties. More precisely, X_t is said to be α-stable if

$$X_1 \stackrel{d}{=} \frac{1}{t^{1/\alpha}} X_t \qquad \text{for all } t > 0.$$

These processes are the equivalent to Gaussian random processes when dealing with infinite variance random variables. Indeed, the Generalized Central Limit Theorem states that, under certain assumptions, the distribution of the sum of infinite variance random variables converges to a stable distribution (see for example [32] for a precise statement of this result).

Stable processes can be used to model real-world phenomena [32,20], and in particular they are commonly used in Mathematical Finance; see for example [26,11,27–29,8] and references therein.

The infinitesimal generator of any symmetric stable Lévy process is of the form

$$Lu(x) = \int_{S^{n-1}} \int_{-\infty}^{+\infty} \big(u(x+\theta r) + u(x-\theta r) - 2u(x)\big) \frac{dr}{|r|^{1+2s}}\, d\mu(\theta), \tag{1.1}$$

where μ is any nonnegative and finite measure on the unit sphere, called the *spectral measure*, and $s \in (0, 1)$.

The aim of this paper is to establish new and sharp interior and boundary regularity results for general symmetric stable operators (1.1).

Remarkably, the only ellipticity assumptions in all our results will be

$$0 < \lambda \le \inf_{\nu \in S^{n-1}} \int_{S^{n-1}} |\nu \cdot \theta|^{2s} d\mu(\theta), \qquad \int_{S^{n-1}} d\mu \le \Lambda < \infty. \tag{1.2}$$

Notice that these hypotheses are satisfied for *any* symmetric stable operator whose spectral measure μ is n-dimensional, i.e., such that there is no hyperplane V of $\mathbb{R}^n$ such that μ is supported on V. Notice also that in case that the spectral measure μ is supported on an hyperplane V, then no regularity result holds.

When the spectral measure is absolutely continuous, $d\mu(\theta) = a(\theta)d\theta$, then these operators can be written as

$$Lu(x) = \int_{\mathbb{R}^n} \big(u(x+y) + u(x-y) - 2u(x)\big) \frac{a(y/|y|)}{|y|^{n+2s}}\, dy, \tag{1.3}$$

where $a \in L^1(S^{n-1})$ is a nonnegative and even function.

The most simple example of stable Lévy process X_t in $\mathbb{R}^n$ is the one corresponding to $d\mu(\theta) = c\, d\theta$, with $c > 0$. In this case, the operator L is a multiple of the fractional Lapla-