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The Hardy–Morrey & Hardy–John–Nirenberg inequalities involving distance to the boundary

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Abstract

We strengthen the classical inequality of C.B. Morrey concerning the optimal Hölder continuity of functions in $W^{1,p}$ when $p > n$, by replacing the L^p-modulus of the gradient with the sharp Hardy difference involving distance to the boundary. When $p = n$ we do the same strengthening in the integral form of a well known inequality due to F. John and L. Nirenberg.

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1. Introduction and main results

Let $\Omega \subsetneq \mathbb{R}^n$, $n \geq 1$, be a domain and denote the distance function to its boundary $\partial\Omega$ by

$$d(x) := \inf_{y \in \partial\Omega} |x - y|, \quad \text{whenever } x \in \bar{\Omega}.$$

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It is proved in [3] that if Ω satisfies the following condition:

$$-\Delta d \geq 0 \text{ in the sense of distributions in } \Omega, \tag{$\mathscr{C}$}$$

then Hardy's inequality holds true with the best possible constant, that is

$$\int_\Omega |\nabla u|^p \mathrm{d}x \geq \Big(\frac{p-1}{p}\Big)^p \int_\Omega \frac{|u|^p}{d^p} \mathrm{d}x \quad \text{for all } u \in C_c^\infty(\Omega), \tag{1.1}$$

where $p > 1$ is arbitrary. Examples of domains satisfying condition ($\mathscr{C}$) are convex domains since then d is superharmonic in Ω (see [2]). Moreover, if the boundary $\partial\Omega$ is smooth enough, say uniformly of class C^2 (see Definition 2.3), then ($\mathscr{C}$) is known to be *equivalent* to the domain being mean convex, i.e. having nonnegative mean curvature everywhere on its boundary (see [25] and also [14], [19] and [12]). In view of this, we call *weakly mean convex* domain any domain satisfying condition ($\mathscr{C}$).

For $p = 2$ and Ω being the half-space, i.e. $\Omega = \mathbb{R}^n_+$ where

$$\mathbb{R}^n_+ := \{(x', x_n) \mid x' = (x_1, ..., x_{n-1}) \in \mathbb{R}^{n-1},\ x_n > 0\},\ n \geq 2,$$

the critical Sobolev norm can be added on the right hand side of (1.1). More precisely, Maz'ya in his treatise [22] proved that for $n \geq 3$ there exists a positive constant C such that

$$\Bigg(\int_{\mathbb{R}^n_+} |\nabla u|^2 \mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^n_+} \frac{u^2}{x_n^2}\mathrm{d}x\Bigg)^{1/2} \geq C\Bigg(\int_{\mathbb{R}^n_+} |u|^{2^*}\mathrm{d}x\Bigg)^{1/2^*} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n_+), \tag{1.2}$$

where $2^* := 2n/(n-2)$. This inequality has been extended to domains in [9]. It is proved there that if Ω is a uniformly C^2 mean convex domain with finite inner radius, that is

$$D_\Omega := \sup_{x\in\Omega} d(x) < \infty,$$

then there exists a positive constant C such that

$$\Bigg(\int_\Omega |\nabla u|^2 \mathrm{d}x - \frac{1}{4}\int_\Omega \frac{|u|^2}{d^2}\mathrm{d}x\Bigg)^{1/2} \geq C\Bigg(\int_\Omega |u|^{2^*}\mathrm{d}x\Bigg)^{1/2^*} \quad \text{for all } u \in C_c^\infty(\Omega). \tag{1.3}$$

It is also known (see [11]) that if one strengthens assumption ($\mathscr{C}$) to convexity, then (1.3) holds true with a constant C independent of the domain Ω and without any regularity assumption on Ω.

At this point we want to compare the above result with the corresponding result for Hardy's inequality with the distance taken from a point in Ω. It is known (see [10, Theorem A] and also [1]) that if Ω is a bounded domain containing the origin, then there exists a positive constant C such that for any $u \in C_c^\infty(\Omega)$ the following estimate holds true

$$\Bigg(\int_\Omega |\nabla u|^2 \mathrm{d}x - \Big(\frac{n-2}{2}\Big)^2\int_\Omega \frac{|u|^2}{|x|^2}\mathrm{d}x\Bigg)^2 \geq C\Bigg(\int_\Omega |u|^{2^*} X^{1+2^*/2}\Big(\frac{|x|}{R_\Omega}\Big)\mathrm{d}x\Bigg)^{1/2^*}. \tag{1.4}$$