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Optimal lower bound for the first eigenvalue of the fourth order equation

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Abstract

In this paper we will find optimal lower bound for the first eigenvalue of the fourth order equation with integrable potentials when the L^1 norm of potentials is known. We establish the minimization characterization for the first eigenvalue of the measure differential equation, which plays an important role in the extremal problem of ordinary differential equation. The conclusion of this paper will illustrate a new and very interesting phenomenon that the minimizing measures will no longer be located at the center of the interval when the norm is large enough.

MSC: 34L15; 34L40

Keywords: Eigenvalue; The fourth order equation; Integrable potential; Minimization problem; Optimal lower bound

1. Introduction

Given an integrable potential $q \in \mathcal{L}^1 := \mathcal{L}^1([0,1],\mathbb{R})$, we consider eigenvalue problem of the fourth order beam equation

$$y^{(4)}(t) + q(t)y(t) = \lambda y(t), \qquad t \in [0,1], \tag{1.1}$$

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with the Lidstone boundary condition

$$y(0) = y''(0) = 0 = y(1) = y''(1). \tag{1.2}$$

It is well-known that problem (1.1), (1.2) has a sequence of (real) eigenvalues

$$\lambda_1(q) < \lambda_2(q) < \cdots < \lambda_m(q) < \cdots,$$

satisfying $\lim_{m\to\infty} \lambda_m(q) = +\infty$. See [3]. For constant potentials, one has

$$\lambda_m(c) = (m\pi)^4 + c \qquad \forall m \in \mathbb{N}, \quad c \in \mathbb{R}. \tag{1.3}$$

In this paper we are concerned with the first eigenvalues $\lambda_1(q)$ and will give their optimal lower bounds when the L^1 norms $\|q\|_1 = \|q\|_{L^1([0,1])}$ are known. To this end, we will solve the following minimization problem

$$\mathbf{L}(r) := \inf\{\lambda_1(q) : q \in B_1[r]\}. \tag{1.4}$$

Here, for $r \in [0, +\infty)$,

$$B_1[r] := \left\{ q \in \mathcal{L}^1 : \|q\|_1 \leq r \right\}$$

is the ball of $(\mathcal{L}^1, \|\cdot\|_1)$. Once minimization problem (1.4) is solved, one has the following lower bound for $\lambda_1(q)$

$$\lambda_1(q) \geq \mathbf{L}(\|q\|_1) \qquad \forall q \in \mathcal{L}^1, \tag{1.5}$$

which will be shown to be optimal in a certain sense.

Problems linking the coefficient of an operator to the sequence of its eigenvalues are among the most fascinating of mathematical analysis. One of the reasons which make them so attractive is that the solutions are involved of many different branches of mathematics. Moreover, they are very simple to state and generally hard to solve. For both ordinary and partial differential operators, there have evolved a lot of results [1,5–7,9,14,19,23]. In recent years, the authors and their collaborators have revealed some deep properties on the dependence of eigenvalues on potentials and completely solved the extremal value problems for eigenvalues of the second order Sturm–Liouville and p-Laplace operators with potentials varied in balls in $\mathcal{L}^1$ space. See [11,18,22,24]. These extremal value problems cannot be solved directly by variational methods, because eigenvalues $\lambda_n(q)$ are implicit functionals of potentials q, the space $\mathcal{L}^1$ is infinite dimensional, and the balls $B_1[r]$ with radius r in $\mathcal{L}^1$ are non-compact non-smooth sets. Based on some topological facts on Lebesgue spaces and strong continuity and Fréchet differentiability of eigenvalues in potentials, the authors have used an analytical method to solve these extremal value problems in two steps.

Step 1. Deal with the corresponding problems in $\mathcal{L}^p$, $p > 1$, space by using the standard variational methods, because those balls $B_p[r]$ in L^p are smooth in usual topology and compact in weak topology when $p > 1$. Obtained in this step are the critical equations, which are autonomous Hamiltonian systems of 1-degree-of-freedom.