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On the Burgers–Poisson equation

K. Grunert [a,*], Khai T. Nguyen [b]

[a] *Department of Mathematical Sciences, NTNU, Norwegian University of Science and Technology, NO-7491 Trondheim, Norway*

[b] *Department of Mathematics, Penn State University, PA 16802, USA*

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Abstract

In this paper, we prove the existence and uniqueness of weak entropy solutions to the Burgers–Poisson equation for initial data in $\mathbf{L}^1(\mathbb{R})$. In addition an Oleinik type estimate is established and some criteria on local smoothness and wave breaking for weak entropy solutions are provided.

MSC: primary 35L03; secondary 35L67

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1. Introduction

The Burgers–Poisson equation is given by the balance law obtained from Burgers' equation by adding a nonlocal source term

$$u_t + \left(\frac{u^2}{2}\right)_x = [G * u]_x, \tag{1.1}$$

* Corresponding author.
E-mail addresses: katring@math.ntnu.no (K. Grunert), ktn2@psu.edu (K.T. Nguyen).
URLs: http://www.math.ntnu.no/~katring/ (K. Grunert), http://www.personal.psu.edu/ktn2/ (K.T. Nguyen).

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where

$$G(x) = -\frac{1}{2}e^{-|x|} \quad \text{and} \quad [G * u](x) = -\frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-z|} \cdot u(z)dz$$

solve the Poisson equation

$$\varphi_{xx} - \varphi = u.$$

Equation (1.1) has been derived in [9] as a simplified model of shallow water waves and admits conservation of both momentum and energy. For sufficiently smooth initial data

$$u(0, x) = \overline{u}(x), \tag{1.2}$$

the local existence and uniqueness of solutions of (1.1) have been established in [5]. In addition the analysis of traveling waves showed that the equation features wave breaking in finite time. Hence it is natural to study existence and uniqueness of weak entropy solutions which are global in time.

Definition 1.1. A function $u \in \mathbf{L}^1_{loc}([0, \infty[\times \mathbb{R}) \cap \mathbf{L}^\infty_{loc}(]0, \infty[, \mathbf{L}^\infty(\mathbb{R}))$ is a **weak entropy solution** of (1.1)–(1.2) if u satisfies the following properties:

(i) The map $t \mapsto u(t, \cdot)$ is continuous with values in $\mathbf{L}^1(\mathbb{R})$ and satisfies the initial condition (1.2).
(ii) For any $k \in \mathbb{R}$ and any non-negative test function $\phi \in C^1_c(]0, \infty[\times \mathbb{R}, \mathbb{R})$ one has

$$\int\int \Big[|u - k|\phi_t + \text{sign}(u - k)\Big(\frac{u^2}{2} - \frac{k^2}{2}\Big)\phi_x + \text{sign}(u - k)[G_x * u(t, \cdot)](x)\phi\Big] dxdt \geq 0. \tag{1.3}$$

For any initial data $\bar{u} \in BV(\mathbb{R})$, the existence of a global weak entropy solution to (1.1)–(1.2) has been studied in [5]. The proof is based on the vanishing viscosity method yielding a sequence of approximating smooth solutions. Due to the BV bound of $\bar{u}$, one obtains that the approximating solutions also satisfy an a priori uniform BV bound for all positive times, yielding the compactness of the approximating sequence of solutions. However this method cannot be applied in the more general case with initial data in $\mathbf{L}^1(\mathbb{R})$. In addition there are no uniqueness or continuity results for global weak entropy solutions of (1.1) established in [5]. Thus our main goal is to study the existence and uniqueness for global weak entropy solutions for initial data $\bar{u} \in \mathbf{L}^1(\mathbb{R})$. To be more explicit we are going to show the following theorem.

Theorem 1.2. *Given any initial data $u(0, \cdot) = \bar{u}(\cdot) \in \mathbf{L}^1(\mathbb{R})$, the Cauchy problem (1.1)–(1.2) has a unique weak entropy solution $u(t, x)$ in $[0, \infty) \times \mathbb{R}$. Furthermore, for any $t > 0$*

$$\|u(t, \cdot)\|_{\mathbf{L}^1(\mathbb{R})} \leq e^t \, \|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}, \tag{1.4}$$