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# Global and nonglobal existence for a strongly coupled parabolic system on a general domain

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**Abstract**

We consider the parabolic system $u_t - \Delta u = f(t)u^r v^s$, $v_t - \Delta v = g(t)u^q v^s$, in $\Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^N$ is either an unbounded or bounded domain and $f, g \in C[0, \infty)$. We find conditions for the global existence or nonglobal existence, which are expressed in terms of the behavior of $\|S(t)u_0\|_\infty$ as $t \to \infty$, where $u(t) = S(t)u_0$ is the solution of the linear problem $u_t - \Delta u = 0$, $u(0) = u_0$.


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## 1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be any domain with smooth boundary $\partial\Omega$ and $T > 0$. We consider the following coupled parabolic system

$$\begin{cases} u_t - \Delta u = f(t)u^r v^p & \text{in } \Omega \times (0, T), \\ v_t - \Delta v = g(t)u^q v^s & \text{in } \Omega \times (0, T), \\ u = v = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 \geq 0, \ \ v(0) = v_0 \geq 0 & \text{in } \Omega, \end{cases} \tag{1}$$

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where $u_0, v_0 \in C_0(\Omega)$, $r, s, p, q > 0$ and $f, g \in C[0, \infty)$.

Systems of the form (1) naturally arise in studying nonlinear phenomena in biology, chemistry and physics. For instance, system (1) has been used to model densities and temperatures in chemical reactions, condensate amplitudes in Bose–Einstein condensates, wave amplitudes (or envelops of multiple interacting optical modes) in optical fibers, and pattern formation in ecological systems. The quantities $u$, $v$ represent densities, temperatures, amplitudes, etc., and are nonnegative, see [18] and references therein.

It is well known that problem (1) admits a solution, defined on a maximal interval $[0, T_{max})$, $(u, v) \in C([0, T_{max}), [C_0(\Omega)]^2)$ verifying

$$\begin{cases} u(t) = S(t)u_0 + \displaystyle\int_0^t S(t-\sigma) f(\sigma) u^r(\sigma) v^p(\sigma) d\sigma, \\ v(t) = S(t)v_0 + \displaystyle\int_0^t S(t-\sigma) g(\sigma) u^q(\sigma) v^s(\sigma) d\sigma, \end{cases} \tag{2}$$

for every $t \in [0, T_{max})$. Moreover, we have that either $T_{max} = +\infty$ (global solution) or $T_{max} < \infty$ and $\limsup_{t \to T_{max}} (\|u(t)\|_\infty + \|v(t)\|_\infty) = +\infty$ (nonglobal solution or blow up in finite time solution), see for instance [3,5,7] and the references therein. Henceforth, $(S(t))_{t \geq 0}$ denotes the heat semigroup with the Dirichlet condition on the boundary.

When $\Omega$ is a bounded domain and $f = g \equiv 1$, problem (1) was considered in [3]. We denote $\varphi_1$ as the first eigenfunction of the Laplacian operator associated to the first eigenvalue $\lambda_1 > 0$ in $H_0^1(\Omega)$. We assume that $\int_\Omega \varphi_1(x) dx = 1$.

**Theorem 1.1** *([3]). Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $f = g \equiv 1$, $p, q, r, s > 0$ and*

$$D = (1-r)(1-s) - pq. \tag{3}$$

(i) *If $r > 1$ or $s > 1$ or $D < 0$, then problem (1) admits both global and nonglobal solution. Moreover,*
  (a) *If $u_0 \geq C\varphi_1, v_0 \geq C\varphi_1$ for $C > 0$ sufficiently large then the solution of problem (1) is nonglobal.*
  (b) *If $u_0 \leq \varphi^a, v_0 \leq \varphi^b$ with $a$ and $b$ suitably chosen, then the solution of problem (1) is global.*

(ii) *If $r < 1$, $s < 1$ and $D \geq 0$, then all solutions of problem (1) are global.*

The situation is more delicate when $\Omega$ is the whole space.

**Theorem 1.2** *([7]). Let $\Omega = \mathbb{R}^N$, $f = g \equiv 1$, $pq > 0$ and $r + p \leq q + s$. Assume that $(u, v)$ is not of the form $(u, 0)$ or $(0, v)$ so that $u > 0$ and $v > 0$ on $(0, T)$.*

(i) *Suppose that $r > 1$.*
  (a) *If $(r + p - 1)^{-1} < N/2$, then problem (1) has both global and nonglobal solutions.*
  (b) *If $(r + p - 1)^{-1} \geq N/2$, then every nontrivial solution of problem (1) is nonglobal.*