Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 261 (2016) 3670–3703

Journal of Differential Equations

www.elsevier.com/locate/jde

Existence and uniqueness of solutions for Navier–Stokes equations with hyper-dissipation in a large space

Zhijie Nan [a,b], Xiaoxin Zheng [a,*]

[a] *School of Mathematics and Systems Science, Beihang University, Beijing 100191, PR China*
[b] *Department of Mathematics, Physics and Information Engineering, Nanhu College, Jiaxing University, Zhejiang 314001, PR China*

Received 15 August 2015; revised 27 March 2016

Available online 28 June 2016

Abstract

We study Cauchy problem of the 3D Navier–Stokes equations with hyper-dissipation. By using the Fourier localization technique, we prove that the system has a unique global solution for large initial data in a critical Fourier–Herz space. More importantly, the energy of this solution is infinite.

Keywords: Navier–Stokes equations; Fourier–Herz space; Infinite energy solution; Global well-posedness; Kato's algorithm

1. Introduction

In this paper, we consider Cauchy problem for the incompressible Navier–Stokes equation with fractional dissipation:

$$\begin{cases} \partial_t u + \Lambda^{2\alpha} u + (u \cdot \nabla) u + \nabla P = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

* Corresponding author.
E-mail addresses: nanzhijie@163.com (Z. Nan), xiaoxinzheng@buaa.edu.cn (X. Zheng).

http://dx.doi.org/10.1016/j.jde.2016.05.035
0022-0396/

where $\Lambda \triangleq (-\Delta)^{\frac{1}{2}}$, and $u = u(x,t)$, $P = P(x,t)$ denote velocity and pressure of the fluid, respectively. In addition, the initial datum u_0 satisfies $\nabla \cdot u_0 = 0$.

Problem (1.1) has attracted much attention and some important results were obtained in the past years. Let us begin with the weak solutions of system (1.1). The incompressible condition that $\nabla \cdot u = 0$ enables us to get the following natural energy

$$E(u)(t) \triangleq \frac{1}{2}\int_{\mathbb{R}^n} |u(x,t)|^2\,dx + \int_0^t \int_{\mathbb{R}^n} \left|\Lambda^\alpha u(x,\tau)\right|^2 dxd\tau \le \frac{1}{2}\int_{\mathbb{R}^n} |u_0(x)|^2\,dx. \tag{1.2}$$

This natural energy estimate together with the compactness argument entails the global-in-time existence of weak solutions u. On the other hand, we easily find that the solution u of system (1.1) enjoys the scaling property. More exactly, if u is a solution to (1.1) associated with the data u_0, so are the rescaled functions

$$u_\lambda(x,t) \triangleq \lambda^{2\alpha-1}u(\lambda x, \lambda^{2\alpha}t) \quad \text{and} \quad u_{0,\lambda}(x) \triangleq \lambda^{2\alpha-1}u_0(\lambda x). \tag{1.3}$$

A simple computation yields

$$E(u_\lambda) = \lambda^{4\alpha-2-n}E(u).$$

The scaling property implies that the case $\alpha < \frac{n+2}{4}$ is supercritical, the case $\alpha = \frac{n+2}{4}$ is critical and the case $\alpha > \frac{n+2}{4}$ is subcritical. This implies that it is difficult to use the L^2 energy to control the nonlinear term. Thus, the uniqueness of weak solution of supercritical case is still an open problem. When $\alpha = 1$, problem (1.1) becomes the classical Navier–Stokes equations, which have been widely studied. The first important result was obtained by Leray in his seminal paper [14]. He proved that finite energy initial data generates global solutions. Nevertheless, the uniqueness of such solutions remains open besides that space dimension is two. Also, there are some results on the global well-posedness for small initial data in several critical spaces such as the homogeneous Sobolev space $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, the Lebesgue space $L^3(\mathbb{R}^3)$ and so on. Here the critical space X is a translation invariant Banach space of tempered distributions satisfying $\|u_\lambda\|_X = \|u\|_X$. Fujita and Kato [9] gave a global well-posedness result for small initial data in some critical spaces such as the homogeneous Sobolev space $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. Later on, Kato [13] showed the same theorem in the Lebesgue space $L^3(\mathbb{R}^3)$. After that, several results were established in the homogeneous Besov spaces $\dot{B}_{p,\infty}^{-1+\frac{3}{p}}(\mathbb{R}^3)$ [5,6]. Recently, a global existence theorem was given by Koch and Tataru [12] in the space $BMO^{-1}(\mathbb{R}^3)$. This result seems optimal in some sense, because ill-posedness is observed in larger spaces such as $\dot{B}_{\infty,q}^{-1}$ with $q > 2$, see for example [3,11,18].

In [10], Gallagher and Planchon follow Calderón's procedure [4] and perform an interpolation between Leray's weak solutions and Kato's mild solutions. As the interpolation space must be between $L^2(\mathbb{R}^2)$ and $BMO^{-1}(\mathbb{R}^2)$, the Besov spaces $\dot{B}_{p,q}^{-1+\frac{2}{p}}(\mathbb{R}^2)$ $(p > 2)$ which have high-frequency oscillation data appear naturally.

When $\alpha \ge \frac{5}{4}$, it is well-known that (1.1) has a global strong solution for $n = 3$ provided $u_0 \in L^2(\mathbb{R}^3)$. In this article, our goal is to prove that the following system (1.4) exists as a unique global solution which enjoys infinite energy