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The energy balance relation for weak solutions of the density-dependent Navier–Stokes equations ☆

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Abstract

We consider the incompressible inhomogeneous Navier–Stokes equations with constant viscosity coefficient and density which is bounded and bounded away from zero. We show that the energy balance relation for this system holds for weak solutions if the velocity, density, and pressure belong to a range of Besov spaces of smoothness $1/3$. A density-dependent version of the classical Kármán–Howarth–Monin relation is derived.

MSC: 76S05; 35Q35

Keywords: Navier–Stokes equation; Onsager conjecture; Kármán–Howarth–Monin relation; Turbulence

1. Introduction

Consider the density-dependent incompressible Navier–Stokes equations:

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu \Delta u = -\nabla p + \rho f, \tag{1}$$

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{2}$$

$$\nabla \cdot u = 0. \tag{3}$$

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Here $u(x,t)$ represents the d-dimensional velocity, $f(x,t)$ is an external force (with values in $\mathbb{R}^d$), $p(x,t)$ is the pressure, $\rho(x,t)$ is the density, and μ is the viscosity coefficient (which we take to be constant). We consider (1)–(3) for $x \in \mathbb{T}^d$ and $t \geq 0$. It is known, see [11,10,13], that if u_0 is divergence-free and square-integrable, $\underline{\rho} \leq \rho_0 \leq \overline{\rho}$ for some positive constants $\underline{\rho}$ and $\overline{\rho}$, and if $f \in L^2([0,T]; L^2(\mathbb{T}^d))$, then there exists a Leray–Hopf type global weak solution to the system (ρ, u) such that $\underline{\rho} \leq \rho \leq \overline{\rho}$, $u \in L^2([0,T]; H^1(\mathbb{T}^d))$, and (ρ, u) satisfies the energy inequality

$$E(t) - E(0) \leq -\mu \int_0^t \|\nabla u\|^2_{L^2(\mathbb{T}^d)}\, ds + \int_0^t \int_{\mathbb{T}^d} \rho u \cdot f\, dx\, ds, \qquad \text{where } E(s) = \frac{1}{2} \int_{\mathbb{T}^d \times \{s\}} \rho |u|^2\, dx. \tag{4}$$

Fluids with variable distribution of density arise in many physical contexts. In particular, they appear prominently in Rayleigh–Taylor mixing when a heavier layer fluid on top of lighter one gets mixed under the force of gravity, creating a non-homogeneous turbulent layer. Although an analogue of the classical Kolmogorov theory of turbulence for non-homogeneous fluids has not yet been developed, it appears to be evident that under proper self-similarity assumptions on the velocity increments $\delta u = u(r+\ell) - u(r)$ and density $\delta\rho$ a limited level of regularity would be expected of u and ρ in the limit of vanishing viscosity. Such regularity should allow for a residual amount of energy to be dissipated in the limit by analogy with the Kolmogorov's 0th law of turbulence, see [9]. Mathematical study of the question of what this critical regularity might be has been a subject of many recent publications centered around the so-called Onsager conjecture, which states that for the pure Euler equation Hölder exponent $1/3$ gives a threshold regularity between energy conservation and existence of dissipative solutions that do not conserve energy (see [6,4,2,12,1,5]). In this paper we address the same question in the context of the full density-dependent forced system (1)–(3) with or without viscosity.

Let us recall that a weak solution to (1)–(3) is a triple $(\rho, u, p) \in L^\infty_{t,x} \times L^2_{t,x} \times \mathcal{D}'$ ($\mathcal{D}'$ is the space of distributions) such that for any triple of smooth test functions (η, ψ, γ), one has

$$\begin{aligned} \int_{\mathbb{T}^d \times \{s\}} \rho u \cdot \psi\, dx \Big|_0^t - \int_0^t \int_{\mathbb{T}^d} \big(\rho u \cdot \partial_s \psi + (\rho u \otimes u) : \nabla \psi + p \operatorname{div} \psi\big)\, dx\, ds \\ = \mu \int_0^t \int_{\mathbb{T}^d} u \cdot \Delta \psi\, dx\, ds + \int_0^t \int_{\mathbb{T}^d} \rho f \cdot \psi\, dx\, ds, \end{aligned} \tag{5}$$

$$\int_{\mathbb{T}^d \times \{s\}} \rho \eta\, dx \Big|_0^t = \int_0^t \int_{\mathbb{T}^d} (\rho \partial_s \eta + (\rho u \cdot \nabla)\eta)\, dx\, ds, \tag{6}$$

$$\int_{\mathbb{T}^d} u \cdot \nabla \gamma = 0. \tag{7}$$