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J. Differential Equations 261 (2016) 3864–3892

Journal of Differential Equations

www.elsevier.com/locate/jde

Interaction of modulated gravity water waves of finite depth

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Received 31 January 2016; revised 1 May 2016

Available online 4 July 2016

Abstract

We consider the capillary-gravity water wave problem of finite depth with a flat bottom of one or two horizontal dimensions. We derive the modulation equations of leading and next-to-leading order in the hyperbolic scaling for three weakly amplitude-modulated plane wave solutions of the linearized problem in the absence of quadratic and cubic resonances. We justify the derived system of macroscopic equations in the case of gravity waves using the stability of the finite depth water wave problem on the time scale $O(1/\epsilon)$.

MSC: primary 76B15; secondary 76B45, 35B35, 35L03, 35L05

Keywords: water wave problem of finite depth; interaction of waves; justification of modulation equations

1. Introduction

The capillary-gravity water wave problem of finite depth can be written in the following non-dimensionalized form, due to Zakharov [41], Craig, C. Sulem, P.-L. Sulem [8,9,33], and Alvarez-Samaniego, Lannes [3,25]:

$$\partial_t U + \mathcal{N}_{\epsilon,\sigma}(U) = 0, \qquad U = (\zeta, \psi)^T, \qquad \mathcal{N}_{\epsilon,\sigma} = (\mathcal{N}^1_{\epsilon,\sigma}, \mathcal{N}^2_{\epsilon,\sigma})^T, \tag{1.1}$$

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http://dx.doi.org/10.1016/j.jde.2016.06.011

where

$$\mathcal{N}^1_{\epsilon,\sigma}(U) = -\mathcal{G}[\epsilon\zeta]\psi, \tag{1.2}$$

$$\mathcal{N}^2_{\epsilon,\sigma}(U) = \zeta - \tfrac{1}{\mathrm{Bo}}\nabla\cdot\Big(\frac{\nabla\zeta}{\sqrt{1+\epsilon^2|\nabla\zeta|^2}}\Big) + \frac{\epsilon}{2}|\nabla\psi|^2 - \frac{\epsilon}{2}\frac{(\mathcal{G}[\epsilon\zeta]\psi + \epsilon\nabla\zeta\cdot\nabla\psi)^2}{1+\epsilon^2|\nabla\zeta|^2}. \tag{1.3}$$

Here, the unknown functions of time $t \in [0, T)$, $T > 0$, and horizontal space $X = (x, y) \in \mathbb{R}^d$, $d = 1, 2$, are the free surface elevation $\epsilon\zeta : [0, T) \times \mathbb{R}^d \to \mathbb{R}$ above the still water level $z = 0$ and the trace $\psi : [0, T) \times \mathbb{R}^d \to \mathbb{R}$ of the velocity potential Φ at $z = \epsilon\zeta$ of the water beneath this surface and above the fixed flat bottom $z = -\sqrt{\mu}$, $\mu \in (0, \infty)$, extending over all of $\mathbb{R}^d$. The scaling parameter $0 < \epsilon \leqslant 1$ is the steepness of the wave, i.e., the ratio of the amplitude of $\epsilon\zeta$ to the characteristic horizontal length $L = 1$. The second term in (1.3) corresponds to the surface tension, which is essentially the mean curvature of the surface scaled by the (inverse) Bond number $\frac{1}{\mathrm{Bo}} = \frac{\sigma}{\rho g}$, where σ, ρ, g are the (dimensionless) coefficients of the surface tension, the fluid density and the gravity acceleration, respectively. When $\sigma = 0$, this term is absent and we speak of gravity water waves.

The most important term in the water wave equation (1.1) is the Dirichlet–Neumann operator

$$\mathcal{G}[\epsilon\zeta]\psi = \partial_z\Phi(\cdot, \epsilon\zeta) - \nabla(\epsilon\zeta)\cdot\nabla\Phi(\cdot, \epsilon\zeta) = \sqrt{1 + |\nabla(\epsilon\zeta)|^2}\,\partial_{\mathbf{n}}\Phi(\cdot, \epsilon\zeta), \tag{1.4}$$

where the velocity potential Φ of the fluid is the unique solution of the boundary value problem for the Laplace equation

$$\begin{cases} \Delta_{X,z}\Phi = 0, & -\sqrt{\mu} \leqslant z \leqslant \epsilon\zeta, \\ \Phi(\cdot, \epsilon\zeta) = \psi, & \partial_z\Phi(\cdot, -\sqrt{\mu}) = 0 \end{cases} \tag{1.5}$$

in the fluid domain at time $t \in [0, T)$ with Dirichlet data ψ at the surface and Neumann boundary data at the bottom. With $\mathbf{n}$ denoting the upward unit normal vector at the surface $\epsilon\zeta$, we see that the Dirichlet–Neumann operator (1.4) relates the Dirichlet data ψ to the normal derivative of the potential Φ at the surface $\epsilon\zeta$, and that it is linear in ψ but nonlinear in ζ.

For a detailed derivation of the water wave problem and an overview of the progress in the analytical proof of its well-posedness, as well as the various asymptotic limits derived for different settings and scalings, we refer the reader to [25] and the references given therein. We restrict here ourselves to mentioning only the breakthrough local and global well-posedness results [37–40] of Wu and [14] of Germain, Masmoudi, Shatah for gravity waves of infinite depth, and the local well-posedness result of Alvarez-Samaniego, Lannes [24,3,4] in the case of finite depth and on time-scales of order $O(1/\epsilon)$. In the case of capillary-gravity waves we mention exemplarily the work of Iguchi [18], Ambrose, Masmoudi [5], Ming, Zhang [27], and Alazard, Burq, Zuily [1], and refer to these articles for further references.

Among the various asymptotic limits that can be derived from the water wave problem (1.1), one particular class concerns modulation equations. Based on the nonlinear and dispersive character of the problem in the case of deep water, one can consider (small) amplitude modulations of plane wave solutions to the linearization of (1.1) around $(\zeta, \psi) = (0, 0)$. It turns out, that the linearized problem