Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 261 (2016) 3893–3940

Journal of Differential Equations

www.elsevier.com/locate/jde

Remarks on the asymptotic behavior of the solution to damped wave equations

Hisashi Nishiyama

Department of Education, Wakayama University, 930 Sakaedani, Wakayama city, Wakayama 640-8510, Japan

Received 2 October 2015; revised 21 April 2016

Available online 28 June 2016

Abstract

We study the diffusion phenomena for damped wave equations. We prove that the solution of an abstract damped wave equation becomes closer to the solution of a heat type equation as time tends to infinity under some assumption to the resolvent. As an application of our approach, we also study the asymptotic behavior of the damped wave equation in Euclidean space under the geometric control condition.

MSC: 35L05; 35K05; 35Q99; 46N20

Keywords: Damped wave equation; Heat equation; Diffusion phenomena; Geometric control condition; Variable damping; Energy decay

1. Introduction

We study the following damped wave equation on $\mathbf{R}^d$

$$\left(\partial_t^2 - \sum_{i,j=1}^{d} \partial_i g_{ij}(x)\partial_j + a(x)\partial_t\right) u(x,t) = (\partial_t^2 + P + a(x)\partial_t)u(x,t) = 0. \tag{1.1}$$

E-mail address: h2480@center.wakayama-u.ac.jp.

http://dx.doi.org/10.1016/j.jde.2016.06.014
0022-0396/

Here we write $\partial_t = \frac{\partial}{\partial t}$, $\partial_i = \frac{\partial}{\partial x_i}$ and $P = -\sum_{i,j=1}^{d} \partial_i g_{ij}(x)\partial_j$. $g_{ij}(\cdot)$ and $a(\cdot)$ are in $\mathcal{B}^\infty(\mathbf{R}^d)$; the space of smooth functions with bounded derivatives. We assume $(g_{ij}(\cdot))$ is a family of uniformly elliptic real symmetric matrices, i.e. there exists a constant $C > 0$ such that

$$\frac{1}{C}\mathrm{Id} \le (g_{ij}(x)) \le C\mathrm{Id}$$

holds for any $x \in \mathbf{R}^d$. Under this assumption, P is a positive definite self-adjoint operator on H^2 and $D(\sqrt{P}) = H^1$ where H^1 and H^2 are usual Sobolev spaces on $\mathbf{R}^d$. We also assume $a(x) \ge 0$ for any $x \in \mathbf{R}^d$.

In this paper, we shall study the case that $a(x)$ may vanish however the geometric control condition holds. So we recall about the geometric control condition. For the operator P, we define its Hamiltonian flow $\phi_t(x,\xi) = (x(t), \xi(t))$ by the solution of the following Hamilton's equation

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{\partial p}{\partial \xi}, \quad \dfrac{d\xi}{dt} = -\dfrac{\partial p}{\partial x}, \\ (x(0), \xi(0)) = (x, \xi). \end{cases}$$

Here the symbol $p(x,\xi) = \sum_{i,j=1}^{d} g_{ij}(x)\xi_i\xi_j$ is associated with P. By the uniform ellipticity, the above Hamilton's equation has time global solutions for any initial data and we can define the Hamiltonian flow. For the damped wave equation, it is known that the mean value of the damping term along a solution of Hamilton's equation is closely related to the asymptotic profile of the solution cf. [7,20]. So we introduce

$$\langle a \rangle_T(x,\xi) = \frac{1}{T}\int_0^T a(\phi_t(x,\xi))dt$$

where $T > 0$ and we use the notion $a(x,\xi) := a(x)$. We study the asymptotic behavior of the damped wave equation under the following geometric control condition.

Geometric control condition: There exist $T > 0$ and $\alpha > 0$ such that for all $(x,\xi) \in p^{-1}(\{1\})$, we have $\langle a \rangle_T(x,\xi) \ge \alpha$.

The following result is a uniform energy decay under the geometric control condition.

Theorem 1.1. *We assume the geometric control condition. Let u be a solution of (1.1) with the initial data $u|_{t=0} = u_0 \in H^1$ and $\partial_t u|_{t=0} = u_1 \in L^2$. Then there exists a constant C such that for $t > 1$, the following bound holds*

$$\|\nabla u\|_{L^2}^2 + \|\partial_t u\|_{L^2}^2 \le C\Big(\sum_{i,j=1}^{d} (g_{ij}\partial_i u, \partial_j u)_{L^2} + \|\partial_t u\|_{L^2}^2\Big) \le Ct^{-1}(\|u_0\|_{H^1}^2 + \|u_1\|_{L^2}^2).$$

Remark 1.2. The above bound is optimal without any further assumption to the initial data. Moreover it is known that the geometric control condition is necessary for the uniform energy decay from $H^1 \times L^2$ norm of the initial data.