Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 261 (2016) 3941–3963

Journal of Differential Equations

www.elsevier.com/locate/jde

Pointwise nonlinear stability of nonlocalized modulated periodic reaction–diffusion waves

Soyeun Jung [a,1], Kevin Zumbrun [b]

[a] *Kongju National University, Gongju-si, Chungcheongnamdo 314-701, Republic of Korea*
[b] *Indiana University, Bloomington, IN 47405, USA*

Received 7 May 2016

Available online 8 July 2016

Abstract

In this paper, extending previous results of [2], we obtain pointwise nonlinear stability of periodic traveling reaction–diffusion waves, assuming spectral linearized stability, under nonlocalized perturbations. More precisely, we establish pointwise estimate of nonlocalized modulational perturbation under a small initial perturbation consisting of a nonlocalized modulation plus a localized perturbation decaying algebraically.

1. Introduction

We consider a system of reaction–diffusion equations

$$u_t = u_{xx} + f(u), \tag{1.1}$$

where $(x,t) \in \mathbb{R} \times \mathbb{R}^+$, $u \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ sufficiently smooth. We assume that $u(x,t) = \bar{u}(x - ct)$ is a traveling wave solution of the system (1.1) with a constant speed c and the profile $\bar{u}(\cdot)$ satisfies $\bar{u}(\cdot) = \bar{u}(\cdot + 1)$. In other words, $\bar{u}(x)$ is a stationary 1-periodic solution of the PDE

E-mail addresses: soyjung@kongju.ac.kr (S. Jung), kzumbrun@indiana.edu (K. Zumbrun).
[1] This work was supported by the research grant of the Kongju National University in 2015.

http://dx.doi.org/10.1016/j.jde.2016.06.013

$$u_t = u_{xx} + cu_x + f(u). \tag{1.2}$$

In [2], the first author established pointwise Green function bounds on the linearized operator about the underlying solution $\bar{u}$ and obtained pointwise nonlinear stability of $\bar{u}$ by estimating the localized modulational perturbation $v(x,t) = \tilde{u}(x - \psi(x,t), t) - \bar{u}(x)$ $(h_0 = \psi(x,0) = 0)$ under small initial perturbations $v(x,0) = \tilde{u}(x,0) - \bar{u}(x)$ decaying algebraically for nearby solutions $\tilde{u}$ of (1.2).

In the present paper, we study the pointwise nonlinear stability of $\bar{u}$ of (1.2) under small perturbations consisting of a nonlocalized modulation ($h_0(x) = \psi(x,0)$ does not decay algebraically, but $\partial_x h_0$ decays algebraically) plus a localized perturbation $v(x,0) = \tilde{u}(x - h_0(x), 0) - \bar{u}(x)$ ($v(x,0)$ decays algebraically). Johnson, Noble, Rodrigues and Zumbrun showed L^p-nonlinear stability under such nonlocalized modulational perturbations ($h_0 \notin L^1$, but $\partial_x h_0 \in L^1$) for systems of reaction–diffusion equations in [4] and of conservation laws in [6]. Sandstede, Scheel, Schneider, and Uecker obtained similar results by rather different methods for systems of reaction–diffusion equations in [10].

Similarly as in [4,6], here, we determine an appropriate nonlocalized modulation $\psi(x,t)$ by an adaption of the basic nonlinear iteration scheme developed in [7]. However, in the absence of cancellation estimates afforded by Hausdorff–Young and Parseval inequalities, we find it necessary to decompose the solution a bit differently than was done in [4] in order to estimate sharply the key "modulation" part of the linearized solution operator in response to modulational-type data (see Remark 3.2), and to estimate this modulational part essentially "by hand." This is the main new difficulty in our analysis beyond those carried out in [2,4].

1.1. Preliminaries

We first linearize the PDE (1.2) about a stationary 1-periodic solution $\bar{u}$ so that we obtain the eigenvalue problem

$$\lambda v = Lv := (\partial_x^2 + c\partial_x + df(\bar{u}))v, \tag{1.3}$$

operating on $L^2(\mathbb{R})$ with densely defined domains $H^2(\mathbb{R})$. Here, v is considered as a perturbation of $\bar{u}$ defined by $v(x,t) = \tilde{u}(x,t) - \bar{u}(x)$ for nearby solutions $\tilde{u}$. To characterize the $L^2(\mathbb{R})$-spectrum of L (denoted by $\sigma_{L^2(\mathbb{R})}(L)$), we rewrite (1.3) as the following linear ODE system

$$V_x = \mathbb{A}V, \quad \text{where } V = \begin{pmatrix} v \\ v_x \end{pmatrix} \text{ and } \mathbb{A} = \mathbb{A}(x,\lambda) = \begin{pmatrix} 0 & I \\ \lambda I - df(\bar{u}) & -cI \end{pmatrix}. \tag{1.4}$$

Since all coefficients of L are 1-periodic, $\mathbb{A}(x+1,\lambda) = \mathbb{A}(x,\lambda)$; so by Floquet theory, the fundamental matrix solution $\Phi(x,\lambda)$ of the linear system (1.4) is

$$\Phi(x,\lambda) = P(x,\lambda)e^{R(\lambda)x},$$

where $R(\lambda) \in \mathbb{C}^{n\times n}$ is a constant matrix and $P(x,\lambda) \in \mathbb{C}^{n\times n}$ is a periodic matrix, $P(x,\lambda) = P(x+1,\lambda)$. In fact, for each eigenvalue μ (referred to as the Floquet exponent) of $R(\lambda)$, there is a solution to (1.4) of the form $V(x,\lambda) = e^{\mu x}W(x,\lambda)$, where W is 1-periodic in x. Thus, any nontrivial solution V to the system (1.4) does not lie in $L^2(\mathbb{R})$, which means that the $L^2(\mathbb{R})$-spectrum