Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 261 (2016) 4008–4029
Journal of Differential Equations
www.elsevier.com/locate/jde

# On piecewise smooth vector fields tangent to nested tori

Tiago Carvalho [a,*], Marco A. Teixeira [b,c]

[a] *FC–UNESP, 17033-360, Bauru, São Paulo, Brazil*
[b] *IMECC–UNICAMP, 13081-970, Campinas, São Paulo, Brazil*
[c] *UFSCar-campus Sorocaba, CEP 18052-780, Sorocaba, São Paulo, Brazil*

Received 3 February 2016; revised 13 May 2016
Available online 27 June 2016

**Abstract**

In this paper we present a number of results involving 3D nonsmooth dynamical systems tangent to a foliation. We study one-parameter families of systems $Z_\varepsilon$ passing through a specific model $Z_0$ whose phase portrait is foliated by invariant nested tori. For each positive integer $k$ we, explicitly, construct a family $Z_\varepsilon^k$ possessing exactly $k$ nested tori.


*MSC:* 34A36; 34C05; 37G15; 37G35

*Keywords:* Nonsmooth vector field; Invariant tori; Limit cycles; Bifurcation

## 1. Introduction

### 1.1. The prelude

Over the last decades there has been much progress in understanding complex phenomena in the qualitative theory of differential equations through the study of smooth dynamical systems. In particular, the development of theories that take into account essential symmetries coming from integrability or conservative properties of dynamical systems has been very effective in

* Corresponding author.
*E-mail addresses:* tcarvalho@fc.unesp.br (T. Carvalho), teixeira@ime.unicamp.br (M.A. Teixeira).

http://dx.doi.org/10.1016/j.jde.2016.06.015
0022-0396/

providing a theoretical basis for complicated phenomena such as pattern formation and symmetry breaking problems. The emphasis in these studies has been mainly on spatial properties of smooth systems.

Integrability in nonsmooth systems was originally studied separately (and less actively, at least theoretically) despite its relevance and appearance in many applications of general interest, including Control Theory, Impact phenomena and also in piecewise smooth Hamiltonian mechanics. We refer to [14] and references therein for many illustrative models in this direction.

A consistent comparison between smooth, continuous nonsmooth and discontinuous nonsmooth systems could be a leading theme of research not only for the beauty of mathematical results, but also because of the huge applicability of the results in practical problems. We are interested in the particular subject: perturbations of systems possessing first integral or more general, vector fields tangent to foliations. Mainly, a systematic program towards the bifurcation theory for such systems and problems involving persistence of invariant tori are currently emergent. In this direction, the model studied in the present paper is, as long as the authors know, the first initiative to establish in 3D a discussion involving the late question for nonsmooth systems. As such subject is still poorly understood in higher dimensions, we hope that the present article will serve as a starting point to clarify such intriguing field.

### 1.2. Goal

The amount of publications on Piecewise Smooth Vector Fields (PSVFs, for short) and Piecewise Maps is vast (see [1,2,4,14,16] and references therein) and they often appear as models for a plenty of phenomena.

Despite of the fact that some recent papers (see [6,7,13,15]) analyze the dynamical behavior of the flow of PSVFs with tangencies at the separation boundaries and their bifurcations, as well as the authors know, in the literature there is no a qualitative and systematic study of nonsmooth vector fields tangent to foliations. In this paper we present a number of results involving bifurcation and persistence of invariant topological tori when an *"integrable"* nonsmooth dynamical systems defined in $\mathbb{R}^3$ is perturbed.

Observe that we deal with nonsmooth vector fields in $\mathbb{R}^3$ where the switching set is concentrated in a codimension one submanifold of $\mathbb{R}^3$. As usual the definition of how trajectories of $Z$ behave through points of the switching manifold is given by the Filippov convention (see [10] and Section 2 below).

We analyze small perturbations of a PSVF $Z_0$ in $\mathbb{R}^3$ defined by:

$$Z_0(x,y,z) = \begin{cases} X(x,y,z) = \left(x,\, y,\, \frac{4+(x^2+y^2)(17-12(x^2+y^2))}{4\sqrt{x^2+y^2}}\right) & \text{if } z \geq 0, \\ Y(x,y,z) = \left(-x,\, -y,\, \frac{4+(x^2+y^2)(17-12(x^2+y^2))}{4\sqrt{x^2+y^2}}\right) & \text{if } z \leq 0. \end{cases} \tag{1}$$

In this case we use the notation $Z_0 = (X, Y)$. Observe that $Z_0$ is such that $Z_0(x,y,z) = Z_0(-x,-y,-z)$ and so, it is *$\varphi$-reversible*, where $\varphi(x,y,z) = (-x,-y,-z)$.

Notice that the mathematical model represented by $Z_0$, did not originate from a practical phenomenon. Its origin is purely theoretical and it has mainly been inspired on the results of [5]. A non-exhaustive list of future theoretical applications and further directions concerning the subject of this paper is presented in Section 8.