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J. Differential Equations 261 (2016) 4055–4072

Journal of Differential Equations

www.elsevier.com/locate/jde

Splitting scheme for the stability of strong shock profile

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Received 20 January 2016; revised 10 June 2016

Abstract

In this paper, we use the Laplace transform and Dirichlet–Neumann map to give a systematical scheme to study the small wave perturbation of general shock profile with general amplitude. Here we use certain non-classical shock waves for a rotationally invariant system of viscous conservation laws to demonstrate this scheme. We derive an explicit solution and show that it converges pointwise to another over-compressive profile exponentially, when the perturbations of the initial data to a given over-compressive shock profile are sufficiently small.

MSC: 35L65; 35L67

Keywords: Splitting scheme; Strong shock profile; Dirichlet–Neumann map; Conservation law; Nonlinear stability

1. Introduction

Consider the following simple rotationally invariant system originated from the study of MHD and nonlinear elasticity by Freistühler [3],

$$\begin{cases} \tilde{u}_t + (\tilde{u}(\tilde{u}^2 + \tilde{v}^2))_x = \mu \tilde{u}_{xx}, \\ \tilde{v}_t + (\tilde{v}(\tilde{u}^2 + \tilde{v}^2))_x = \mu \tilde{v}_{xx}. \end{cases} \tag{1.1}$$

The characteristics are

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http://dx.doi.org/10.1016/j.jde.2016.06.018

$$r_1(\tilde u,\tilde v)=(\tilde v,-\tilde u),\quad r_2(\tilde u,\tilde v)=(\tilde u,\tilde v),\quad \lambda_1(\tilde u,\tilde v)=\tilde u^2+\tilde v^2,\quad \lambda_2(\tilde u,\tilde v)=3\lambda_1(\tilde u,\tilde v). \tag{1.2}$$

The 1-characteristic is linearly degenerate and the 2-characteristic is genuinely nonlinear except at the origin

$$\nabla\lambda_1\cdot r_1(\tilde u,\tilde v)=0,\quad \nabla\lambda_2\cdot r_2(\tilde u,\tilde v)=6(\tilde u^2+\tilde v^2). \tag{1.3}$$

A viscous shock wave has end states along the same radial direction through the origin, i.e., in the direction of $r_2(\tilde u,\tilde v)$. The system is rotational invariant and so, without loss of generality, consider the end states to have the second component zero $(\tilde u_\pm, 0)$. When $\tilde u_\pm$ are of the same sign, the shock is classical, and there is only one connecting orbit along the $\tilde u_-$ axis. When $\tilde u_+ < 0$, the shock may cross the non-strictly hyperbolic point $(\tilde u,\tilde v)=0$ and becomes over-compressive.

Here we are interested in over-compressive shock, which is characterized by

$$\lambda_2(\tilde u_-,\tilde v_-)>\lambda_1(\tilde u_-,\tilde v_-)>b>\lambda_2(\tilde u_+,\tilde v_+)>\lambda_1(\tilde u_+,\tilde v_+), \tag{1.4}$$

b is the speed of over-compressive shock. (1.4) implies that the over-compressive shock is a node–node connection. Thus when exists, there is a 1-parameter family of viscous profiles. More details about the over-compressive shock will be given in the Section 2.

The goal of this paper is to prove that the over-compressive shock profiles for (1.1) can be stable against small perturbations using our new approach: given the profile $\Phi=(\phi,\psi)$ of an (appropriate) over-compressive shock profile, and a function $(\tilde u_0(x),\tilde v_0(x))$ (of appropriate type) such that

$$\begin{pmatrix} u_0(x)\\ v_0(x)\end{pmatrix}\equiv\begin{pmatrix}\tilde u_0(x)\\ \tilde v_0(x)\end{pmatrix}-\Phi \tag{1.5}$$

is small (in an appropriate sense), then the solution $(\tilde u,\tilde v)$ of (1.1) with initial data $(\tilde u_0(x),\tilde v_0(x))$ converges time-asymptotically to another profile $\Phi^*=(\phi^*,\psi^*)$. We give the pointwise convergence rate to the new profile.

Since there is a 1-parameter family of viscous shocks with given end states, the stability of the shock profile is to be understood in the following way: the perturbation of a stable shock profile would converge to the same or another profile in the 1-parameter family. Thus, in addition to the phase shift, the perturbation also changes the time-asymptotic profile of the solution. Therefore, instead of using the conservation laws to identify the phase shift and diffusion waves as for the Laxian shocks, we use the two conservation laws to identify the phase shift and the new profile. This should make the situation well-posed as we have two conservation laws and the same number of parameters to determine. Here and thereafter, we set $\mu=1$ without loss of generality.

The main theorem is given as follows:

Theorem 1.1 *(Main Theorem). Give an over-compressive shock profile $\Phi=(\phi,0)$ with the shock speed b. Let C be a universal positive constant, and let the initial perturbation defined by (1.5) satisfy $|u_0(x),v_0(x)|\le O(1)\varepsilon e^{-|x|/C}$. Then for ε sufficiently small, there exists a unique profile Φ^* with $\Phi(\pm\infty)=\Phi^*(\pm\infty)$, such that the solution of (1.1) satisfies:*