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The Moore–Gibson–Thompson equation with memory in the critical case

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Abstract

We consider the following abstract version of the Moore–Gibson–Thompson equation with memory

$$\partial_{ttt}u(t) + \alpha\partial_{tt}u(t) + \beta A\partial_t u(t) + \gamma Au(t) - \int_0^t g(s)Au(t-s)\mathrm{d}s = 0$$

depending on the parameters $\alpha, \beta, \gamma > 0$, where A is strictly positive selfadjoint linear operator and g is a convex (nonnegative) memory kernel. In the subcritical case $\alpha\beta > \gamma$, the related energy has been shown to decay exponentially in [19]. Here we discuss the critical case $\alpha\beta = \gamma$, and we prove that exponential stability occurs if and only if A is a bounded operator. Nonetheless, the energy decays to zero when A is unbounded as well.

MSC: 35B35; 35G05; 45D05; 47D06

Keywords: Moore–Gibson–Thompson equation with memory; Memory kernel; Exponential decay; Semiuniform stability

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1. Introduction

Let $(\mathrm{H}, \langle\cdot,\cdot\rangle, \|\cdot\|)$ be a separable real Hilbert space, and let

$$A : \mathrm{H} \to \mathrm{H} \qquad \text{of domain} \qquad \mathfrak{D}(A) \subset \mathrm{H}$$

be a strictly positive selfadjoint linear operator, where the (dense) embedding $\mathfrak{D}(A) \subset \mathrm{H}$ need not be compact. In this work we consider the integrodifferential equation

$$\partial_{ttt}u(t) + \alpha\partial_{tt}u(t) + \beta A\partial_t u(t) + \gamma Au(t) - \int_0^t g(s)Au(t-s)\mathrm{d}s = 0. \tag{1.1}$$

Here α, β, γ are strictly positive constants subject to the further constraint

$$\varkappa := \beta - \frac{\gamma}{\alpha} \geq 0,$$

while the (nonnull) function $g \in W^{1,1}(\mathbb{R}^+)$ with g' absolutely continuous on $\mathbb{R}^+ = (0, \infty)$, usually called *memory kernel*, satisfies the following assumptions:

(g1) $g(s) \geq 0$ and $g'(s) \leq 0$ for every $s > 0$.
(g2) $g'' \geq 0$ almost everywhere.
(g3) $\varrho := \int_0^\infty g(s)\mathrm{d}s \in (0, \gamma)$.
(g4) There exists $\delta > 0$ such that $g'(s) + \delta g(s) \leq 0$ for every $s > 0$.

The equation is supplemented with the initial conditions assigned at time $t = 0$

$$\begin{cases} u(0) = u_0, \\ \partial_t u(0) = v_0, \\ \partial_{tt}u(0) = w_0, \end{cases}$$

being u_0, v_0, w_0 prescribed initial data.

1.1. Physical motivations and background

The integrodifferential equation (1.1) is a natural outgrowth of the Moore–Gibson–Thompson (MGT) equation arising in nonlinear acoustics, and accounting for the second sound effects and the associated thermal relaxation in viscous gases/fluids [10,24,32]. We also address the reader to the work of Stokes [30], where presumably the MGT equation appeared for the first time. Introducing additional nonlocal effects due to molecular relaxation, one arrives at the viscoelastic version (1.1) of the MGT equation [11,21,25].

In order to gain a better understanding of the MGT equation, we shall begin with the Westervelt (or more generally Kuznetsov) equation, one of the fundamental models for nonlinear acoustic waves (see [12,14,33] and references therein). Denoting by β the diffusivity coefficient and letting $\gamma = c^2$ (the square of the sound speed), the Westervelt equation written for the state variable $u = u(t)$, standing for acoustic pressure, is given by