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On fractional matrix exponentials and their explicit calculation

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Abstract

Fractional matrix exponentials are introduced, which extend the usual matrix exponential involving ordinary derivatives to the case of fractional derivative operators. Two fractional analogues are defined, corresponding to the Caputo and Riemann–Liouville fractional derivatives. Moreover, explicit methods similar to Putzer's method for calculating the usual matrix exponential are developed for these fractional matrix exponentials.

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1. Introduction

Many natural processes can be modeled by systems of linear, first-order, ordinary differential equations (ODEs) of the form

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{1.1}$$

where $t \geqslant 0$ usually represents time, $\mathbf{A}$ is an $n \times n$ real (or complex) matrix, $\mathbf{x}(t)$ is an $n \times 1$ vector that represents the state of the system at time t, and $\mathbf{x}_0$ is an $n \times 1$ vector that prescribes

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the initial state. It is well known that the unique solution of the initial-value problem (IVP) given in (1.1) is

$$\mathbf{x}(t) = \exp(t\mathbf{A})\mathbf{x}_0,$$

where $\exp(t\mathbf{A})$ is the matrix exponential defined formally as

$$\exp(t\mathbf{A}) = \mathbf{I} + \frac{t}{1!}\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \cdots = \sum_{j=0}^{\infty} \frac{t^j}{j!}\mathbf{A}^j$$

and $\mathbf{I}$ is the $n \times n$ identity matrix. Thus the solution of the IVP (1.1) reduces to the calculation of a matrix exponential.

If $\mathbf{D} = \operatorname{diag}(d_1, \ldots, d_n)$ is a diagonal matrix, then

$$\exp(t\mathbf{D}) = \operatorname{diag}(e^{d_1 t}, \ldots, e^{d_n t}).$$

Furthermore, if $\mathbf{A}$ is diagonalizable, i.e., $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D} = \operatorname{diag}(d_1, \ldots, d_n)$ for some invertible $n \times n$ matrix $\mathbf{P}$, then

$$\exp(t\mathbf{A}) = \mathbf{P}\exp(t\mathbf{D})\mathbf{P}^{-1}.$$

More generally, if $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{J}$, where $\mathbf{J}$ is the Jordan canonical form of an arbitrary matrix $\mathbf{A}$, then the calculation of $\exp(t\mathbf{A})$ is equivalent to the calculation of $\exp(t\mathbf{J})$, which is not always straightforward.

Putzer [25] proposed an algorithm for calculating the matrix exponential that avoids the use of the Jordan canonical form but requires the solution of a recursive system of linear ODEs. The procedure is as follows. Let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of $\mathbf{A}$. Define the matrices $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_n$ by

$$\mathbf{P}_0 = \mathbf{I}, \quad \mathbf{P}_k = (\mathbf{A} - \lambda_k\mathbf{I}) \cdots (\mathbf{A} - \lambda_1\mathbf{I}), \quad k = 1, \ldots, n.$$

It follows from the Cayley–Hamilton theorem that $\mathbf{P}_n = \mathbf{O}$, where $\mathbf{O}$ is the $n \times n$ zero matrix. Let $y_1, \ldots, y_n$ be n functions of t that satisfy the IVP

$$\begin{aligned} &y_1'(t) = \lambda_1 y_1(t), \quad y_1(0) = 1, \\ &y_{k+1}'(t) = \lambda_{k+1} y_{k+1}(t) + y_k(t), \quad y_{k+1}(0) = 0, \quad k = 1, \ldots, n-1. \end{aligned}$$

Then a finite expansion of the matrix exponential is

$$\exp(t\mathbf{A}) = \sum_{k=0}^{n-1} y_{k+1}(t)\mathbf{P}_k.$$

Putzer's method is generic in the sense that it can be applied to any square matrix and even with repeated eigenvalues.

It is easily verified that the matrix-valued function $\mathbf{\Phi}(t) = \exp(t\mathbf{A})$ satisfies the IVP

$$\mathbf{\Phi}'(t) = \mathbf{A}\mathbf{\Phi}(t), \quad \mathbf{\Phi}(0) = \mathbf{I}.$$