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Viscous singular shock profiles for a system of conservation laws modeling two-phase flow

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Abstract

This paper is concerned with singular shocks for a system of conservation laws via the Dafermos regularization $u_t + f(u)_x = \epsilon t u_{xx}$. For a system modeling incompressible two-phase fluid flow, the existence of viscous profiles is proved using Geometric Singular Perturbation Theory. The weak convergence and the growth rate of the viscous solution are also derived; the weak limit is the sum of a piecewise constant function and a δ-measure supported on a shock line, and the maximum value of the viscous solution is of order $\exp(1/\epsilon)$.

MSC: 35L65; 35L67; 34E15; 34C37

Keywords: Conservation laws; Singular shocks; Viscous profiles; Dafermos regularization; Geometric Singular Perturbation Theory

1. Introduction

Keyfitz et al. [22,23] considered the system of conservation laws

$$
\begin{aligned}
\beta_t + (vB_1(\beta))_x &= 0 \\
v_t + (v^2 B_2(\beta))_x &= 0
\end{aligned}
\tag{1}
$$

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where $t \geq 0$, $x \in \mathbb{R}$, $v \in \mathbb{R}$, $\beta \in [\rho_1, \rho_2]$ with $0 < \rho_2 < \rho_1$ and

$$B_1(\beta) = \frac{(\beta - \rho_1)(\beta - \rho_2)}{\beta}, \quad B_2(\beta) = \frac{\beta^2 - \rho_1\rho_2}{2\beta^2}. \tag{2}$$

For Riemann problems with data in feasible regions, they constructed uniquely defined admissible solutions. It can be readily shown that this system is not everywhere hyperbolic, and hence standard methods do not apply (see e.g. [6,37]). To resolve this problem, along with rarefaction waves and regular shocks, the concept of singular shocks was adopted. A singular shock solution, roughly speaking, is a distribution which contains delta measures and is the weak limit of a sequence of approximate viscous solutions. For details of the definition, we refer to [21,35].

The existence of singular shocks for (1) was proved in [23]. Their proof was based on a construction of approximate solution of the regularized system via Dafermos regularization

$$\begin{aligned} &\beta_t + (vB_1(\beta))_x = \epsilon t \beta_{xx} \\ &v_t + (v^2 B_2(\beta))_x = \epsilon t v_{xx} \end{aligned} \tag{3ϵ}$$

for certain Riemann data

$$(\beta, v)(x, 0) = \begin{cases} (\beta_L, v_L), & x < 0 \\ (\beta_R, v_R), & x > 0. \end{cases} \tag{4}$$

It was shown that the approximated solutions approach a distribution containing a delta function.

A family of exact solutions of (3ϵ)–(4), rather than approximate solutions, is called a *viscous profile*. In this paper, besides proving existence of viscous profile, we also give descriptions of their limiting behavior, including weak convergence and growth rates. The main tool in our study is *Geometric Singular Perturbation Theory* (GSPT), which will be introduced in Section 4. The use of this tool on singular shocks was first introduced in the pioneering work of Schecter [32].

The system (1) is equivalent to a two-fluid model for incompressible two-phase flow [8, p. 248] of the form

$$\begin{aligned} &\partial_t(\alpha_i) + \partial_x(\alpha_i u_i) = 0 \\ &\partial_t(\alpha_i \rho_i u_i) + \partial_x(\alpha_i \rho_i u_i^2) + \alpha_i \partial_x p_i = F_i, \quad i = 1, 2, \end{aligned} \tag{5}$$

where the drag terms F_i are neglected, the densities ρ_i are assumed to be constant. The six unknowns are reduced to four by the relation $\alpha_1 + \alpha_2 = 1$ and the assumption

$$p_1 = p_2. \tag{6}$$

We follow [22] to replace the volume fractions α_1 and α_2 by a density-weighted volume element $\beta = \rho_2\alpha_1 + \rho_1\alpha_2$ and the momentum equations by a single equation for the momentum difference $v = \rho_1 u_1 - \rho_2 u_2 - (\rho_1 - \rho_2)K$, where $K = \alpha_1 u_1 + \alpha_2 u_2$ is taken to be zero. That is,

$$\alpha_1 = \frac{\rho_1 - \beta}{\rho_1 - \rho_2}, \quad \alpha_2 = \frac{\beta - \rho_2}{\rho_1 - \rho_2}, \quad u_1 = \frac{\beta - \rho_2}{\rho_1 - \rho_2}\frac{v}{\beta}, \quad u_2 = \frac{\beta - \rho_1}{\rho_1 - \rho_2}\frac{v}{\beta}. \tag{7}$$