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Periodic solutions of a perturbed Kepler problem in the plane: From existence to stability

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Abstract

The existence of elliptic periodic solutions of a perturbed Kepler problem is proved. The equations are in the plane and the perturbation depends periodically on time. The proof is based on a local description of the symplectic group in two degrees of freedom.

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1. Introduction

Perturbations of the Kepler problem appear naturally in Celestial Mechanics. These equations are relevant for applications but they also have an intrinsic mathematical interest. In particular the existence and stability of periodic solutions have been discussed by a large number of authors. After Poincaré these questions are usually treated via the averaging method. We refer to the papers [7,11] for results on the autonomous case and for useful lists of references. We will be interested in periodic time dependent perturbations, a class of equations already considered by

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Fatou in [5]. More recently Ambrosetti and Coti Zelati treated in [1] a class of periodic perturbations with symmetries and presented the averaging method in a variational framework.

We are going to consider the perturbed Kepler problem in the plane

$$\ddot{x} = -\frac{x}{|x|^3} + \varepsilon \nabla_x U(t,x), \qquad x \in \mathbb{R}^2 \setminus \{0\}, \tag{1}$$

where ε is a small parameter and U is a smooth function with period 2π in the variable t. In principle U could also depend on ε but we have eliminated this dependence for simplicity. We are interested in the stability properties of 2π-periodic solutions obtained as a continuation from the integrable case $\varepsilon = 0$. These solutions are understood without collisions and so the restriction to an interval of length 2π, say $x : [0, 2\pi] \to \mathbb{R}^2 \setminus \{0\}$, defines a loop in $\mathbb{R}^2 \setminus \{0\}$. In particular, each 2π-periodic solution has a winding number $N \in \mathbb{Z}$. For $\varepsilon = 0$ the system has 2π-periodic solutions with any winding number $N \neq 0$. They are produced by the elliptic (or circular) orbits with major half-axis $a_N = |N|^{-2/3}$. This quantity appears as a consequence of Kepler's third law when we look for solutions with minimal period $\frac{2\pi}{|N|}$. The sign of N corresponds to the orientation of the orbit. Let Σ_N denote the set of initial conditions (x_0, y_0) in $\big(\mathbb{R}^2 \setminus \{0\}\big) \times \mathbb{R}^2$ producing 2π-periodic solution with winding number $N \neq 0$. In view of the relationship between the energy and the major axis of a Keplerian ellipse we can describe Σ_N by the equations

$$\frac{1}{2}|y_0|^2 - \frac{1}{|x_0|} = -\frac{1}{2}N^{2/3}, \qquad N\,(x_{01}y_{02} - x_{02}\,y_{01}) > 0.$$

Later we shall see that Σ_N is a three dimensional manifold diffeomorphic to $\mathbb{S}^1 \times \mathbb{D}$, where $\mathbb{D}$ is the unit open disk.

Let $\phi_t(x_0, y_0) = (x(t; x_0, y_0), y(t; x_0, y_0))$ be the flow associated to the Kepler problem

$$\dot{x} = y, \qquad \dot{y} = -\frac{x}{|x|^3}.$$

Then Σ_N is invariant under ϕ_t and we can average $U(t,x)$ with respect to the flow over the manifold Σ_N to obtain the function

$$\Gamma_N : \Sigma_N \to \mathbb{R}, \qquad \Gamma_N(x_0, y_0) = \frac{1}{2\pi}\int_0^{2\pi} U\,(t, x(t; x_0, y_0))\,dt.$$

Any continuation from $\varepsilon = 0$ of 2π-periodic solutions with winding number N must emanate from the set of critical points of Γ_N. Conversely, if the critical point satisfies some non-degeneracy condition, such a continuation always exists. This type of result can be obtained using an abstract variational framework as in [2] or by a more traditional averaging method. We will follow the second alternative and then it is convenient to employ a system of coordinates which is natural to the equation for $\varepsilon = 0$. Since the Kepler problem is integrable the action-angle variables seem a natural choice (see [6]). These are the well-known Delaunay variables and they work well for the continuation from positive eccentricity. However these coordinates present a blow-up at eccentricity $e = 0$ and they do not seem suitable to deal with the continuation from circular solutions. Poincaré proposed a variant of the Delaunay variables which solves this difficulty (see [9] and [4]). They are of the form $(\lambda, \Lambda, \eta, \xi)$ with $\lambda \in \mathbb{S}^1$, $\Lambda > 0$ and $\eta^2 + \xi^2 < 2\Lambda$.