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L^p gradient estimate for elliptic equations with high-contrast conductivities in $\mathbb{R}^n$

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Abstract

Uniform estimate for the solutions of elliptic equations with high-contrast conductivities in $\mathbb{R}^n$ is concerned. The space domain consists of a periodic connected sub-region and a periodic disconnected matrix block subset. The elliptic equations have fast diffusion in the connected sub-region and slow diffusion in the disconnected subset. Suppose $\epsilon \in (0, 1]$ is the diameter of each matrix block and $\omega^2 \in (0, 1]$ is the conductivity ratio of the disconnected matrix block subset to the connected sub-region. It is proved that the $W^{1,p}$ norm of the elliptic solutions in the connected sub-region is bounded uniformly in ϵ, ω; when $\epsilon \leq \omega$, the L^p norm of the elliptic solutions in the whole space is bounded uniformly in ϵ, ω; the $W^{1,p}$ norm of the elliptic solutions in perforated domains is bounded uniformly in ϵ. However, the L^p norm of the second order derivatives of the solutions in the connected sub-region may not be bounded uniformly in ϵ, ω.

MSC: 35J05; 35J15; 35J25

Keywords: High-contrast conductivity; Potentials; Duality argument; Embedding theory

1. Introduction

Uniform estimate for the solutions of elliptic equations with high-contrast conductivities in $\mathbb{R}^n$ is concerned. The problem has applications in acoustic propagation in porous media, modeling of electromagnetic media, oil recovering process, the stress in composite materials [6,8,14,15,22].

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Suppose $Y \equiv [0,1)^n$ is a cube in $\mathbb{R}^n$ for $n \geq 3$, Y_m is a simply-connected sub-domain of Y with $dist(Y_m, \partial Y) > 0$, $Y_f \equiv Y \setminus \overline{Y_m}$, $\tau \in (0,\infty)$, $\Omega_m^\tau \equiv \{x | x \in \tau(Y_m - j) \text{ for some } j \in \mathbb{Z}^n\}$ is a periodic disconnected subset of $\mathbb{R}^n$, Ω_f^τ $(\equiv \mathbb{R}^n \setminus \overline{\Omega_m^\tau})$ denotes a periodic connected sub-region of $\mathbb{R}^n$, $\partial\Omega_m^\tau$ represents the boundary of Ω_m^τ, and $\mathbf{K}_{\alpha,\tau}(x) \equiv \begin{cases} 1 & \text{if } x \in \Omega_f^\tau \\ \alpha & \text{if } x \in \Omega_m^\tau \end{cases}$ for $\alpha, \tau > 0$. The equation that we consider is

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\omega^2,\epsilon} \nabla\Psi + G) = V & \text{in } \mathbb{R}^n, \\ \lim\limits_{|x|\to\infty} |\Psi|(x) = 0, \end{cases} \tag{1.1}$$

where $\omega, \epsilon \in (0,1]$ and G, V are given functions. If G, V are bounded with compact support, a solution of (1.1) in Hilbert space $\mathcal{D}^{1,2}(\mathbb{R}^n)$ (see definition in section 2) exists uniquely for each ω, ϵ by energy method. The L^2 norm of the gradient of the solution of (1.1) in the connected sub-region Ω_f^ϵ is bounded uniformly in ω, ϵ if G, V are small in Ω_m^ϵ. However, the L^2 norm of the gradient of the solution of (1.1) in matrix blocks Ω_m^ϵ can be very large when ω closes to 0. This is different from the case of uniform elliptic equations, in which uniform bound for $W^{1,p}$ or Lipschitz norm holds in the whole domain [4,5,13,16,21]. To consider nonlinear problems, it is necessary to know whether the uniform bound of the solution of (1.1) in ω, ϵ can be extended to L^p space for any $p \in (1,\infty)$. We note that the L^p estimate of the second derivatives of the solution of (1.1) in the connected sub-region Ω_f^ϵ may not be bounded uniformly in ω, ϵ (see Remark 3.1).

There are some literatures related to this work. Lipschitz estimate and $W^{2,p}$ estimate for uniform elliptic equations with discontinuous coefficients had been proved in [16,21]. Uniform Hölder, $W^{1,p}$, and Lipschitz estimates for uniform elliptic equations with Hölder periodic coefficients were shown in [4,5]. Uniform $W^{1,p}$ estimate for uniform elliptic equations with continuous periodic coefficients was considered in [10] and the same problem with VMO periodic coefficients could be found in [25]. Uniform $W^{1,p}$ estimate for the Laplace equation in periodic perforated domains was considered in [20] and the same problem in Lipschitz estimate was studied in [24]. For non-uniform elliptic equations with smooth periodic coefficients, existence of $C^{2,\alpha}$ solution could be found in [13]. Uniform Hölder and Lipschitz estimates in ω, ϵ for non-uniform elliptic equations with discontinuous periodic coefficients were shown in [27].

In this work, uniform $W^{1,p}$ estimate in ω, ϵ for non-uniform elliptic equations with discontinuous coefficients in $\mathbb{R}^n$ is concerned. We remark that the uniform $W^{1,p}$ estimates for $(1.1)_1$ in bounded domains with Dirichlet boundary conditions are derived in [28], but the constants in the estimates are proportional to the size of the bounded domains. Indeed, the results and the estimate techniques in [28] cannot be used in the problem here. To study the problem, we follow the approach developed by Avellaneda and Lin [5] for uniform (interior) $W^{1,p}$ estimates for elliptic equations with rapidly oscillating, periodic coefficients. Our main effort is the proof of the uniform L^p gradient estimate for the solution of (1.1) for $V = 0$ case. First, we use the compactness argument to establish uniform Hölder and Lipschitz estimates. These estimates are then used to derive uniform bounds as well as error estimates for Green functions and their derivatives. Finally, the L^p gradient estimate is obtained by representing solutions using Green functions and by applying the Potential theory. After the L^p gradient estimate available, the other L^p estimates for the solution of (1.1) can be obtained by using duality argument, Sobolev embedding theory, and extension theory. It is shown that the $W^{1,p}$ norm of the elliptic solutions in the connected