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# Boundedness of solutions for non-linear quasi-periodic differential equations with Liouvillean frequency

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**Abstract**

We study the boundedness of solutions for non-linear quasi-periodic differential equations with Liouvillean frequencies. We proved that if the forcing is quasi-periodic in time with two frequencies which is not super-Liouvillean, then all solutions of the equation are bounded. The proof is based on action-angle variables and modified KAM theory.


## 1. Introduction and main results

### 1.1. Introduction

For the time dependent non-linear differential equation

$$\frac{d^2x}{dt^2} + p(x,t) = 0, \quad x \in \mathbb{R}^1, \tag{1.1}$$

where $p$ is periodic or quasi-periodic in $t$, it is well known that the long time behavior of the solution can be very complicated. For instance, there are equations having unbounded solutions

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but with infinitely many zeros and with nearby unbounded solutions having randomly prescribed number of zeros and also periodic solutions [15]. There are also equations with coexistent unbounded and periodic solutions [1]. On contrary to the unbounded phenomena, Littlewood [10] proposed a question on when all solutions of (1.1) are bounded in the early 1960's.

The problem was studied extensively for the differential equation with superquadratic potentials depending quasi-periodically in time

$$\ddot{x} + x^{2m+1} = \sum_{j=0}^{2m} p_j(\omega t)x^j, \quad x \in \mathbb{R}^1, \tag{1.2}$$

where $\ddot{}$ stands for $\frac{d^2}{dt^2}$, $m \in \mathbb{N}^*$, and $p_j : \mathbb{T}^n \to \mathbb{R}^1$ are real-analytic with the frequency vector $\omega \in \mathbb{T}^n$ being fixed. The first result was due to Morris [13], who proved that all solutions of $\ddot{x} + x^3 = p(t)$ are bounded for continuous and periodic $p$ via Moser's twist theorem [14]. This result was extended to (1.2) with sufficiently smooth periodic $p_j(t)$ by Dieckerhoff and Zehnder [4,5]. Later, their result was extended to more general cases by several authors. We refer to [7,8,11,16,20–25] and references therein. At first glance, it may seem surprising that no smallness assumptions are needed on the time-dependent forcing term, which can change a lot during one period. Yet, these systems turn out to be near-integrable for large amplitude solutions, and thus Moser's twist theorem can be applied to get the boundedness.

When $p_0, p_1, \ldots, p_{2m}$ are quasi-periodic, the above mentioned method failed since the equation (1.2) is no more a time-periodic equation and one cannot apply Moser's twist theorem anymore. Instead, by using the KAM iterations, Levi and Zehnder [9], Liu and You [12] independently proved the Lagrangian stability for (1.2) with $p_0, p_1, \ldots, p_{2m}$ being sufficiently smooth and the frequency $\omega \in \mathbb{T}^n$ being Diophantine

$$|\langle k, \omega \rangle| \geq \frac{\gamma}{|k|^\kappa}, \ \forall\, 0 \neq k \in \mathbb{Z}^n,$$

for some $\gamma > 0, \kappa > n - 1$. One knows that the Diophantine condition is crucial when applying the KAM theory. A natural question is whether the boundedness for all solutions, called Lagrangian stability, still holds if $\omega$ is not Diophantine but Liouvillean? Due to the lack of methods, so far there is no boundedness result if $\omega$ is not Diophantine.

The recent progress in reducibility theory of quasi-periodic linear systems in $sl(2, \mathbb{R})$ of the form

$$\begin{cases} \dot{x} = A(\theta)x, \quad x \in \mathbb{R}^2 \\ \dot{\theta} = \omega = (1, \alpha) \end{cases} \tag{1.3}$$

gives hope for the above mentioned problem. Using KAM approach and Floquet theory, Hou and You [6] proved that if the coefficient matrix $A(\theta)$ is close to a constant matrix, then the system (1.3) is almost reducible and for full Lebesgue measure set of rotation numbers, (1.3) is also rotations reducible, irrespective of any Diophantine condition on $\alpha$. Moreover, Hou and You obtained the non-perturbative reducibility for a full Lebesgue measure set of rotation numbers, provided that $A(\theta)$ is close to a constant ones (the closeness does not depend on $\gamma, \kappa$). Here, almost reducible means that there exists a sequence of quasi-periodic changes of variables, which