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On matrix Painlevé hierarchies

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Abstract

We define a matrix first Painlevé hierarchy and a matrix second Painlevé (P_{II}) hierarchy. For our matrix P_{II} hierarchy we also give auto-Bäcklund transformations and consider the iteration of solutions. This is the first paper to define matrix Painlevé hierarchies and to give auto-Bäcklund transformations for a matrix Painlevé hierarchy. We also consider, amongst other results, the derivation of sequences of special integrals and autonomous limits. Until now it has been unknown how to connect the known matrix P_{II} equation to the obvious candidates for related completely integrable matrix partial differential equations. Our matrix P_{II} hierarchy is placed firmly within the context of a matrix modified Korteweg–de Vries (mKdV) hierarchy. In deriving our matrix P_{II} hierarchy we make use of the Hamiltonian structure of this matrix mKdV hierarchy. We thus see once again the importance for Painlevé hierarchies of the integrability structures of related completely integrable equations.

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1. Introduction

The problem of seeking new transcendental functions defined by ordinary differential equations (ODEs) led, as is well-known, to the discovery of the six Painlevé equations [1–4]. In Painlevé's original classification programme, it was foreseen that after classifying second order ODEs with the Painlevé property, that is, with general solution free of movable branched singularities, third order ODEs would be tackled, then fourth order ODEs, and so on. However, the difficulties of classification increase with the order of the equations studied. This then meant that, in spite of studies of certain classes of third and fourth order ODEs being undertaken [5–10], this classification programme became increasingly difficult to realize.

A partial solution to this problem, though not resulting in a classification as originally foreseen, came about due to the discovery of the connection between completely integrable partial differential equations (PDEs) and ODEs having the Painlevé property [11]. This discovery led to the realization that instead of proceeding "horizontally" (i.e., order by order), it was possible to proceed "vertically" (i.e., by defining a sequence of ODEs of increasing order with properties similar to those of the lowest order member). This was done by Airault [12], and also by Flaschka and Newell [13], who, making use of the fact that sitting above the Korteweg–de Vries (KdV) equation and the modified KdV (mKdV) equation are their respective hierarchies, defined using similarity reduction a hierarchy of ODEs having as first member the second Painlevé (P_{II}) equation. Airault's hierarchy is the now well-known standard P_{II} hierarchy, for every member of which Airault also obtained auto-Bäcklund transformations [12]. Somewhat curiously, however, the idea of "Painlevé hierarchy" lay dormant for some twenty years or so, until Kudryashov's definition of a first Painlevé (P_I) hierarchy and rediscovery of Airault's P_{II} hierarchy [14]. An alternative form of the auto-Bäcklund transformations of this last were given in [15]. Generalized versions of the P_I and P_{II} hierarchies were given in [16]; for the generalized P_{II} hierarchy see also [13]. The connection with completely integrable PDEs, as noted in [11], today informs the classification of higher order ODEs with the Painlevé property; see for example [17,18] and [19,20].

Since the end of the nineties, there has been a great deal of work published on the derivation of Painlevé hierarchies and also of techniques in order to study their properties. For example, in [21], a method was developed of deriving Painlevé hierarchies and their underlying linear problems, by using and extending the observation in [22] that linear problems for Painlevé equations may be obtained from linear problems corresponding to stationary reductions of nonisospectral flows. Further applications of this technique, including to the derivation of discrete and differential-delay Painlevé hierarchies, can be found in [23–27] (see also [28–32]). This approach to deriving Painlevé hierarchies seems more readily applicable than the extension to all members of completely integrable hierarchies of similarity reductions of their first members, an idea that is not necessarily always so easy to put into practice. The use of nonisospectral scattering problems to derive Painlevé hierarchies is discussed further in [33]. It is a nonisospectral approach that we will use in this paper.

A further development has been with regard to the derivation of auto-Bäcklund transformations. In [34] a new method, whose development can be traced back through a series of papers (e.g., [35,36]) to a method of deriving auto-Bäcklund transformations using techniques of singularity analysis [15], was presented. The advantage of this new method, however, since it is phrased in terms of invariances of Miura maps, is that there is no need to study the singularity structure of solutions of the equation. This then means that it can be used not only for ODEs, e.g., [37], but also for other kinds of equation such as discrete equations [34] and differential-