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ScienceDirect
J. Differential Equations 261 (2016) 1203–1236
Journal of Differential Equations
www.elsevier.com/locate/jde

Monotone waves for non-monotone and non-local monostable reaction–diffusion equations

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Received 16 September 2015; revised 2 January 2016

Available online 13 April 2016

Abstract

We propose a new approach for proving existence of monotone wavefronts in non-monotone and non-local monostable diffusive equations. This allows to extend recent results established for the particular case of equations with local delayed reaction. In addition, we demonstrate the uniqueness (modulo translations) of obtained monotone wavefront within the class of all monotone wavefronts (such a kind of conditional uniqueness was recently established for the non-local KPP-Fisher equation by Fang and Zhao). Moreover, we show that if delayed reaction is local then each monotone wavefront is unique (modulo translations) within the class of all non-constant traveling waves. Our approach is based on the construction of suitable fundamental solutions for linear integral-differential equations. We consider two alternative scenarios: in the first one, the fundamental solution is negative (typically holds for the Mackey–Glass diffusive equations) while in the second one, the fundamental solution is non-negative (typically holds for the KPP-Fisher diffusive equations).

MSC: 34K12; 35K57; 92D25

Keywords: Monostable equation; Monotone traveling front; Non-monotone reaction; Non-local interaction; Existence; Uniqueness

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http://dx.doi.org/10.1016/j.jde.2016.03.039
0022-0396/

1. Main results and discussion

Introduction In this work, we study the existence and uniqueness of *monotone* wavefronts $u(x,t) = \phi(x+ct)$ for the monostable delayed non-local reaction–diffusion equation

$$u_t(t,x) = u_{xx}(t,x) - u(t,x) + \int_{\mathbb{R}} K(x-y)g(u(t-h,y))dy, \ u \geq 0, \tag{1}$$

when the reaction term $g : \mathbb{R}_+ \to \mathbb{R}_+$ neither is *monotone* nor defines a *quasi-monotone* functional in the sense of Wu–Zou [46] or Martin–Smith [32] and when the non-negative kernel $K(s)$ is Lebesgue integrable on $\mathbb{R}$. Equation (1) is an important object of studies in the population dynamics, see [2,3,6,15,19,22,29,30,33,34,40,41,44,47–49]. Taking formally $K(s) = \delta(s)$, the Dirac delta function, we obtain the diffusive Mackey–Glass type equation

$$u_t(t,x) = u_{xx}(t,x) - u(t,x) + g(u(t-h,x)), \ u \geq 0, \tag{2}$$

another popular focus of investigation, see [2,21,26,43] for more details and references.

In the classical case, when $h = 0$, all wavefronts to the monostable equation (2) are monotone and, given a fixed admissible wave velocity c, all of them are generated by a unique front by means of translations. The same monotonicity-uniqueness principle is valid for certain subclasses of equations (2) with $h > 0$ (e.g. when g is monotone [43]) and even for equations (1) (e.g. when g is a monotone and globally Lipschitzian function, with the Lipschitz constant $g'(0)$, and when additionally $K(s) = K(-s)$, $s \in \mathbb{R}$ [30,40]). However, if the reaction term is non-local and g is non-monotone, monotonicity and uniqueness are not longer obligatory front's characteristics. For example, [41] provides conditions sufficient for non-monotonicity of wavefronts' profiles for non-local equation (1) with compactly supported kernel K. Co-existence of multiple wavefronts for non-local models is also known from [23,36]. All this explains our interest in establishing effective criteria for the existence and uniqueness of *monotone* wavefronts for the monostable *non-monotone* non-local (or delayed) reaction–diffusion equations. Remarkably, this problem has recently attracted attention of several researchers. In this regard, the most studied model was the non-local KPP-Fisher equation [5,7,16,23,35,36]

$$u_t(t,x) = u_{xx}(t,x) + u(t,x)(1 - \int_{\mathbb{R}} K(x-y)u(t,y)dy), \tag{3}$$

and its local delayed version [5,14,27,24,20,21,46] (called the diffusive Hutchinson's equation)

$$u_t(t,x) = u_{xx}(t,x) + u(t,x)(1 - u(t-\tau,x)). \tag{4}$$

The above cited papers elaborated a complete characterization of models (3) and (4) possessing monotone wavefronts. Moreover, the absolute uniqueness (i.e. uniqueness within the class of all wavefronts) of monotone wavefronts to (4) and the conditional uniqueness (i.e. uniqueness within the subclass of monotone wavefronts) of monotone wavefronts to (3) were also proved in these works. As we have mentioned, in general, monotone and non-monotone wavefronts can coexist in (3) [23,36].

In the case of model (1) having non-monotone function g, the existence of *monotone* wavefronts was analyzed only for the particular case of equation (2) in [21], with the help of the