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Flows for non-smooth vector fields with subexponentially integrable divergence

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Abstract

In this paper, we study flows associated to Sobolev vector fields with subexponentially integrable divergence. Our approach is based on the transport equation following DiPerna–Lions [17]. A key ingredient is to use a quantitative estimate of solutions to the Cauchy problem of transport equation to obtain the regularity of density functions.

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1. Introduction

Since the fundamental work by DiPerna–Lions [17], the study of flows associated to non-smooth vector fields has attracted intensive interest, and has found many applications in PDEs. The problem can be formulated as follows. Given a Sobolev (or more generally BV) vector

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field $b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, does there exist a unique Borel map $X : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, such that

$$\frac{\partial}{\partial t} X(t, x) = b(t, X(t, x)) \tag{1.1}$$

for a.e. $x \in \mathbb{R}^n$? If this ODE is well-posed, then how about the regularity of the solution X?

In the seminal work by DiPerna and Lions [17], the existence of flows for Sobolev velocity fields with bounded divergence was established. Their main ingredient was a careful analysis of the well posedness of the initial value problem for the linear transport equation,

$$\begin{cases} \dfrac{\partial u}{\partial t} + b \cdot \nabla u = 0 & (0, T) \times \mathbb{R}^n, \\ u(0, \cdot) = u_0 & \mathbb{R}^n. \end{cases} \tag{1.2}$$

In their arguments, the notion of renormalized solution was shown to be essential. Later, Ambrosio [1] extended the renormalization property to the setting of bounded variation (BV) vector fields, and obtained the non-smooth flows by using some new tools from Probability and Calculus of Variations. Crippa and De Lellis [14] used a direct approach to recover DiPerna–Lions' theory; see also Bouchut and Crippa [7]. Recently, in [3], Ambrosio, Colombo and Figalli developed a purely local theory on flows for non-smooth vector fields as a natural analogy of the Cauchy–Lipschitz approach. We refer the reader to [2,5,11–13] for more study on transport equations and flows.

Continuing our previous work about the transport equation [10], in this paper we are concerned with the existence of flows for Sobolev vector fields having sub-exponentially integrable divergence. Let us review some developments in this spirit. In [16], Desjardins showed existence and uniqueness of non-smooth flows for velocity fields having exponentially integrable divergence. Later, Cipriano and Cruzeiro [9] analyzed the flows for Sobolev vector fields with exponentially integrable divergence in the setting of Euclidean spaces equipped with Gaussian measures; see [6] for related progresses in Wiener spaces.

As already noticed in [9,6], when the divergence of the velocity field is not bounded, the solution $X(t, \cdot)$ of equation (1.1) still induces a quasi-invariant measure. This motivates the following definition.

Definition 1.1. Let $b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ be a Borel vector field, and $X, \tilde{X} : [0, T] \times [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ be Borel maps.

(i) We say that X is a forward flow associated to b if for each $s \in [0, T]$ and almost every $x \in \mathbb{R}^n$ the map $t \mapsto |b(t, X(s, t, x))|$ belongs to $L^1(s, T)$ and

$$X(s, t, x) = x + \int_s^t b(r, X(s, r, x))\, dr.$$