Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 261 (2016) 1472–1505
Journal of Differential Equations
www.elsevier.com/locate/jde

Population dynamics under selection and mutation: Long-time behavior for differential equations in measure spaces

Azmy S. Ackleh [a,*], John Cleveland [b], Horst R. Thieme [c]

[a] Department of Mathematics, University of Louisiana at Lafayette, Lafayette, LA 70504-1010, USA
[b] Department of Mathematics, University of Wisconsin, Richland Center, WI 53581, USA
[c] School of Mathematical and Statistical Sciences, Arizona State University, Tempe, AZ 85287-1804, USA

Received 6 December 2012; revised 25 March 2016

Available online 26 April 2016

Abstract

We study the long-time behavior of solutions to a measure-valued selection–mutation model that we formulated in [14]. We establish permanence results for the full model, and we study the limiting behavior even when there is more than one strategy of a given fitness; a case that arises in applications. We show that for the pure selection case the solution of the dynamical system converges to a Dirac measure centered at the fittest strategy class provided that the support of the initial measure contains a fittest strategy; thus we term this Dirac measure an Asymptotically Stable Strategy. We also show that when the strategy space is discrete, the selection–mutation model with small mutation has a locally asymptotically stable equilibrium that attracts all initial conditions that are positive at the fittest strategy.

MSC: 91A22; 34G20; 37C25; 92D25

Keywords: Evolutionary game theory; Cone of nonnegative measures; Persistence theory; Survival of the fittest; Asymptotically stable strategy; Lyapunov functions

* Corresponding author.
E-mail address: ackleh@louisiana.edu (A.S. Ackleh).

http://dx.doi.org/10.1016/j.jde.2016.04.008
0022-0396/

1. Introduction

A significant part of evolutionary game theory focuses on the creation and study of mathematical models that describe how the strategy profiles in games change over time due to mutation and selection (replication) [28,38]. In [14] we defined an evolutionary game theory model as an ordered triple $(Q, \mu(t), F(\mu(t)))$ subject to:

$$\frac{d}{dt}\mu(t)(E) = F(\mu(t))(E), \text{ for every } E \in \mathcal{B}(Q). \tag{1}$$

Here Q is the strategy (metric) space, $\mathcal{B}(Q)$ is the σ-algebra of Borel subsets of Q, $\mu(t)$ is a time dependent family of nonnegative finite Borel measures on Q, and F is a density dependent vector field such that μ and F satisfy equation (1). For a Borel subset E of Q, $\mu(t)(E)$ denotes the measure $\mu(t)$ applied to E and can be interpreted as the number of individuals at time t that carry a strategy from the set E.

We also formulated the following selection–mutation evolutionary game theory model as a dynamical system $\phi(t, u, \gamma)$ on the state space of finite nonnegative Borel measures under the weak* topology with $\mu(t) = \phi(t, u, \gamma)$ and $\bar{\mu}(t) = \mu(t)(Q)$:

$$\begin{cases} \dfrac{d}{dt}\mu(t)(E) = \displaystyle\int_Q B\big(\bar{\mu}(t), q\big)\gamma(q)(E)\mu(t)(dq) - \int_E D\big(\bar{\mu}(t), q\big)\mu(t)(dq), \\ \qquad\qquad\;\; = F(\mu(t), \gamma)(E), \\ \quad \mu(0) = u. \end{cases} \tag{2}$$

Here $B(\bar{\mu}(t), q)$ and $D(\bar{\mu}(t), q)$ represent the reproduction (replication) and mortality rates of individuals carrying strategy q when the total population size is $\bar{\mu}(t) = \mu(t)(Q)$. The probability kernel $\gamma(q)(E)$ represents the probability that an individual carrying strategy q produces offspring carrying strategies in the Borel set E. We call γ a *mutation kernel*. We point out that selection–mutation models in the spirit of (2) were formulated as density models on the space of integrable functions and investigated (e.g., see [11–13]).

The purpose of this paper is to complement the well-posedness theory established in [14] with a study of the long-time behavior of solutions to the model (2). In [27], the authors conclude that the eventual outcome of the evolutionary process is characterized by some optimization principle if and only if the environmental feedback affects fitness in an effectively monotone one-dimensional manner. The results in this paper show that, under pure selection evolutionary process, (2) falls under the class of models that can be characterized by an optimization principle: The outcome is a Dirac measure concentrated at the strategy that maximizes the total population size which in turn affects the per capita reproduction and mortality rates in a monotone way. It is well known that the solutions of many such models constructed on the state space of continuous or integrable functions converge to a Dirac measure concentrated at the fittest strategy or trait [2,3,11,12,16,29–31,37].

In [29, Ch. 2], these measure-valued limits are illustrated in a biological and adaptive dynamics environment. This convergence is in the weak* topology [3]. Thus, the asymptotic limit of the solution is not in such state spaces; it is a measure. Some models (e.g. [2,15,20,30,31,37]) have addressed this problem. In [2], the authors formulated a *pure* selection model on the space of finite signed measures with density dependent birth and mortality functions and a 2-dimensional