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Increasing sequences of positive evanescent solutions of nonlinear elliptic equations

Aleksandra Orpel

Faculty of Mathematics and Computer Science, University of Lodz, S. Banacha 22, 90-238 Lodz, Poland

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Abstract

We deal with the existence of a sequence of positive solutions of the following class of nonlinear elliptic problems $\operatorname{div}(a(||x||)\nabla u(x)) + f(x,u(x)) + g(||x||, x \cdot \nabla u(x)) = 0$, where $x \in \mathbb{R}^n$ and $||x|| > R$, with the condition $\lim_{||x|| \to \infty} u(x) = 0$. Our results are based on the iteration approach, in which we apply the subsolution and supersolution method developed by Noussair and Swanson. In this paper we do not assume either growth conditions on f or the equality $f(\cdot, 0) \equiv 0$. We also describe how quickly solutions tend to zero.

1. Introduction

Let us consider the following nonlinear elliptic problem

$$\begin{cases} \operatorname{div}(a(||x||)\nabla u(x)) + f(x,u(x)) + h(||x||, x \cdot \nabla u(x)) = 0, & \text{for } x \in \Omega_R, \\ \lim\limits_{||x|| \to \infty} u(x) = 0, \end{cases} \tag{1}$$

where $n > 2$, $R > 1$, for all $x, y \in \mathbb{R}^n$, $||x|| := \sqrt{\sum_{i=1}^{n} x_i^2}$, and $x \cdot y := \sum_{i=1}^{n} x_i y_i$, $\Omega_R = \{x \in \mathbb{R}^n, ||x|| > R\}$. The goal of this paper is to answer the question when problem (1) possesses

E-mail address: orpela@math.uni.lodz.pl.

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at least k ($k \in N$) classical positive solution and to present the relations among them. We are also interested in conditions which f and h should satisfy in order to guarantee the existence of an increasing (nondecreasing) infinite sequence of different solutions for (1). Moreover we try to characterize more precisely how quickly solutions tend to zero. We want to emphasize that our approach covers nonlinearities f such that for all $u \neq 0$, $f(\cdot, u)$ may change its sign and $f(x, \cdot)$ does not satisfy any growth condition either at 0 or at plus infinity. We have to control only the value of f in a certain neighborhood of zero. In particular our considerations cover the generalized Emden–Fowler equation in a certain exterior domain. The paper is devoted to the case described by the following assumptions

(A_a) *$a : [1, +\infty) \to (0, +\infty)$ belongs to $C^1([1, +\infty))$ and $\lim\limits_{l \to +\infty} a(l) \in (0, +\infty)$;*

(A_f) *$f : \Omega_1 \times \mathbb{R} \to \mathbb{R}$, $\Omega_1 = \{x \in \mathbb{R}^n, ||x|| > 1\}$, is locally Hölder continuous, $x \mapsto f(x, 0)$ is nonnegative in Ω_1 and there exist $d > 0$ and continuous function $M : [1, +\infty) \to (0, +\infty)$ such that $\sup\limits_{u \in [0,d]} \sup\limits_{||x||=r} f(x, u) \le M(r)$ in $[1, +\infty)$ and*

$$\int_1^\infty r^{n-1} M(r)dr < 4(n-2)\, dc\widetilde{a}_{\min}^2, \tag{2}$$

where $c := (n-2)\int_1^\infty \frac{l^{1-n}}{a(l)}dl$, and $\widetilde{a}_{\min} := \inf\limits_{l \in (1,+\infty)} a(l)$;

(A_h1) *$h : [1, +\infty) \times \mathbb{R} \to \mathbb{R}$ is locally Hölder continuous and for every bounded interval $D \subset [1, +\infty)$ there exists $C_D > 0$ such that for all $l \in \overline{D}$ and $m \in R$, $|h(l, m)| \le C_D\left(1 + m^2\right)$;*

(A_h2) *$h(l, 0) = 0$ for all $l \in [1, +\infty)$ and there exists $l_0 \ge 1$, such that $h(l, m) \le 0$ for all $l \ge l_0$ and $m \in [-4d(n-2), 0]$.*

Let us note that assumption (A_f) does not determine the growth of f. This fact allows us to consider sublinear as well as superlinear f, which we shall show in the last section (see Remark 5). The natural examples of h are $h(l, m) := g(l)m$ and $h(l, m) := g(l)m\,(m + 4d(n-2))$, for all $(l, m) \in [1, +\infty) \times \mathbb{R}$, where g is nonnegative in $[l_0, +\infty)$. When we consider the first case for $a \equiv 1$, (1) can be reduced to the following one

$$\begin{cases} \triangle u(x) + f(x, u(x)) + g(||x||)x \cdot \nabla u(x) = 0 \\ \lim\limits_{||x|| \to \infty} u(x) = 0. \end{cases} \tag{3}$$

Such problems have been studied for many years. First of all we have to mention Constantin's results (see e.g. [1–3]), who considered (3) in the case when there exist functions $\widetilde{a} \in C(\mathbb{R}^+, \mathbb{R}^+)$ and w being nonincreasing on $\mathbb{R}^+$ such that

$$0 \le f(x, u) \le M\widetilde{a}(||x||)w(|u|) \text{ for } x \in \mathbb{R}^n \text{ and } u \in \mathbb{R}.$$

In [1], Constantin assumed additionally that $w(s) > 0$ for $s > 0$, $\int_1^\infty \frac{ds}{w(s)} = +\infty$, and g is bounded and satisfies