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Improved conditions for single-point blow-up in reaction–diffusion systems

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Abstract

We study positive blowing-up solutions of the system:

$$u_t - \delta\Delta u = v^p, \quad v_t - \Delta v = u^q,$$

as well as of some more general systems. For any $p, q > 1$, we prove single-point blow-up for any radially decreasing, positive and classical solution in a ball. This improves on previously known results in 3 directions:

(i) no type I blow-up assumption is made (and it is known that this property may fail);

(ii) no equidiffusivity is assumed, i.e. any $\delta > 0$ is allowed;

(iii) a large class of nonlinearities $F(u, v)$, $G(u, v)$ can be handled, which need not follow a precise power behavior.

As a side result, we also obtain lower pointwise estimates for the final blow-up profiles.

MSC: primary 35B40, 35B44, 35B50; secondary 35K61, 35K40, 35K57

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1. Introduction

1.1. Problem and main results

In this paper, we consider nonnegative solutions of the following reaction–diffusion system:

$$\begin{cases} u_t - \delta\Delta u = v^p & x \in \Omega,\ t > 0, \\ v_t - \Delta v = u^q, & x \in \Omega,\ t > 0, \\ u = v = 0, & x \in \partial\Omega,\ t > 0, \\ u(0,x) = u_0(x), & x \in \Omega, \\ v(0,x) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

as well as of the more general system

$$\begin{cases} u_t - \delta\Delta u = F(u,\ v), & x \in \Omega,\ t > 0, \\ v_t - \Delta v = G(u,\ v), & x \in \Omega,\ t > 0, \\ u = v = 0, & x \in \partial\Omega,\ t > 0, \\ u(0,x) = u_0(x), & x \in \Omega, \\ v(0,x) = v_0(x), & x \in \Omega. \end{cases} \tag{1.2}$$

Here $p, q > 1$, $\delta > 0$, $\Omega = B(0,\ R) = \{x \in \mathbb{R}^n\ ;\ |x| < R\}$ with $R > 0$,

$$u_0, v_0 \in L^\infty(\Omega), \quad u_0, v_0 \geq 0, \text{ radially symmetric, radially nonincreasing.} \tag{1.3}$$

As for the functions F and G, we assume that

$$F,\ G \in C^1(\mathbb{R}^2) \tag{1.4}$$

and that system (1.2) is cooperative, i.e.:

$$F_v(u,\ v),\ G_u(u,\ v) \geq 0, \qquad \text{for all } u, v \geq 0. \tag{1.5}$$

Additional assumptions on F, G will be made below.

Under assumptions (1.3)–(1.5), system (1.2) has a unique nonnegative, radially symmetric and radially nonincreasing, maximal classical solution (u, v). This fact follows by standard contraction mapping and maximum principle arguments. The maximal existence time of (u, v) is denoted by $T^* \in (0, \infty]$. If, moreover, $T^* < \infty$, then

$$\limsup_{t\to T^*} (\|u(t)\|_\infty + \|v(t)\|_\infty) = \infty,$$