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Recovering a polyhedral obstacle by a few backscattering measurements

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Abstract

We propose an inverse scattering scheme of recovering a polyhedral obstacle in $\mathbb{R}^n$, $n = 2, 3$, by only a few high-frequency acoustic backscattering measurements. The obstacle could be sound-soft or sound-hard. It is shown that the modulus of the far-field pattern in the backscattering aperture possesses a certain local maximum behavior, from which one can determine the exterior normal directions of the front sides/faces. Then by using the phaseless backscattering data corresponding to a few incident plane waves with suitably chosen incident directions, one can determine the exterior unit normal vector of each side/face of the obstacle. After the determination of the exterior unit normals, the recovery is reduced to a finite-dimensional problem of determining a location point of the obstacle and the distance of each side/face away from the location point. For the latter reconstruction, we need to make use of the far-field data with phases. Numerical experiments are also presented to illustrate the effectiveness of the proposed scheme.

MSC: 78A46; 35Q60

Keywords: Inverse scattering; Polyhedral obstacle; Backscattering; Phaseless

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1. Introduction

This work concerns the inverse scattering problem of recovering an impenetrable obstacle by the corresponding acoustic wave detection. The problem has its physical origin in radar/sonar, geophysical exploration, non-destructive testing and medical imaging (cf. [4,10]). Let D be a bounded Lipschitz domain in $\mathbb{R}^n$, $n = 2, 3$, such that $\mathbb{R}^n\backslash\overline{D}$ is connected. D represents an impenetrable obstacle located in the space and it is assumed to be unknown/inaccessible. In order to identify D, one sends a time-harmonic detecting plane wave of the form

$$u^i(x) = e^{ix\cdot\xi}, \quad \xi \in \mathbb{R}^n, \ \xi \neq 0, \tag{1.1}$$

which is an entire solution to the Helmholtz equation $(-\Delta - |\xi|^2)u = 0$ in $\mathbb{R}^n$. The presence of the obstacle D interrupts the propagation of the plane wave, leading to the so-called scattered wave field u^s, which exists only in the exterior of the obstacle. The total wave field $u = u^i + u^s$ satisfies the following Helmholtz system

$$\begin{aligned} &(-\Delta - |\xi|^2)u = 0 \quad \text{in } \mathbb{R}^n\backslash\overline{D}, \quad \mathcal{B}u = 0 \quad \text{on } \partial D, \\ &\lim_{|x|\to+\infty} |x|^{\frac{n-1}{2}} \left(\frac{\partial u^s}{\partial |x|} - i|\xi|u^s \right) = 0. \end{aligned} \tag{1.2}$$

In (1.2), $\mathcal{B}u := u$ or $\mathcal{B}u := \partial u/\partial\nu$, with $\nu \in \mathbb{S}^{n-1} := \{x \in \mathbb{R}^n; |x| = 1\}$ denoting the exterior unit normal vector to ∂D, corresponding to that D is sound-soft or sound-hard, respectively. In the following, we set $k = |\xi| \in \mathbb{R}_+$ and $d = \xi/|\xi| \in \mathbb{S}^{n-1}$, which denote the wavenumber and incident direction of the plane wave, respectively. The PDE system (1.2) is well-understood with $u \in H^1_{loc}(\mathbb{R}^n\backslash\overline{D})$ possessing the following asymptotic expansion (cf. [4,18])

$$u(x) = e^{ix\cdot\xi} + \frac{e^{ik|x|}}{|x|^{\frac{n-1}{2}}} u^\infty\left(\frac{x}{|x|}\right) + \mathcal{O}\left(\frac{1}{|x|^{\frac{n+1}{2}}}\right), \quad |x| \to +\infty, \tag{1.3}$$

which holds uniformly in $\hat{x} := x/|x| \in \mathbb{S}^{n-1}$, where $x \in \mathbb{R}^n$ and $x \neq 0$. $u^\infty(\hat{x})$ is known as the far-field pattern and we shall write $u^\infty(\hat{x}; \xi, D) = u^\infty(\hat{x}; k, d, D)$ to specify its dependence on the observation direction $\hat{x}$, wavenumber k and incident direction d, as well as the obstacle D. $u^\infty(\hat{x})$ is real-analytic in $\hat{x}$, and hence if it is known on any open patch of $\mathbb{S}^{n-1}$, then it is known on the whole sphere by the analytic continuation; see [4].

The inverse problem that we are concerned with in the present paper is to recover D by knowledge of u^∞, which is known to be nonlinear and ill-posed (cf. [4,10]). It is noted that the inverse problem is formally posed with a fixed $\xi \in \mathbb{R}^n$ and all $\hat{x} \in \mathbb{S}^{n-1}$. Hence, there is a widespread belief that one can recover D by using the far-field pattern corresponding to a single incident plane wave $e^{ix\cdot\xi}$, which is referred to as a single far-field measurement. However, this still remains to be a longstanding problem with very limited progress in the literature. If the obstacle is of small size, roughly speaking, smaller than half of the detecting wavelength, the unique recovery result was established in [5]. Indeed, the study in [5] is based on a spectral argument, and it is shown that if k^2 is less than the first Dirichlet Laplacian eigenvalue of the domain D, then the obstacle can be uniquely determined by a single far-field measurement. If the obstacle is extremely "rough" in the sense that its boundary is nowhere analytic, the unique recovery result was established in [9]. If the obstacle is of general polyhedral type, the corresponding uniqueness study