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J. Differential Equations 260 (2016) 7278–7319

Journal of Differential Equations

www.elsevier.com/locate/jde

Time-dependent singularities in semilinear parabolic equations: Behavior at the singularities

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Received 10 October 2015; revised 18 January 2016

Available online 29 January 2016

Abstract

Singularities of solutions of semilinear parabolic equations are discussed. A typical equation is $\partial_t u - \Delta u = u^p$, $x \in \mathbb{R}^N \setminus \{\xi(t)\}$, $t \in I$. Here $N \geq 2$, $p > 1$, $I \subset \mathbb{R}$ is an open interval and $\xi \in C^\alpha(I; \mathbb{R}^N)$ with $\alpha > 1/2$. For this equation it is shown that every nonnegative solution u satisfies $\partial_t u - \Delta u = u^p + \Lambda$ in $\mathcal{D}'(\mathbb{R}^N \times I)$ for some measure Λ whose support is contained in $\{(\xi(t), t); t \in I\}$. Moreover, if $(N-2)p < N$, then $u(x,t) = (a(t) + o(1))\Psi(x - \xi(t))$ for almost every $t \in I$ as $x \to \xi(t)$, where Ψ is the fundamental solution of Laplace's equation in $\mathbb{R}^N$ and a is some function determined by Λ.

MSC: 35K58; 35A20; 35B60

Keywords: Semilinear parabolic equation; Time-dependent singularity

1. Introduction

This paper is concerned with semilinear parabolic equations including the following problem.

$$\partial_t u - \Delta u = u^p, \quad x \in \Omega \setminus \{\xi(t)\}, \ t \in I. \tag{1.1}$$

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http://dx.doi.org/10.1016/j.jde.2016.01.026
0022-0396/

Here $p > 1$, $\Omega \subset \mathbb{R}^N$ is a smooth domain, $I \subset \mathbb{R}$ is an open interval and $\xi : I \to \Omega$ is a continuous curve. Only in this introduction, we assume that $N \geq 3$ and that ξ is smooth enough. An exceptional point $x = \xi(t)$ in the equation (1.1) is called a time-dependent singularity. In this paper, we are interested in the precise behavior of solutions of (1.1) at the time-dependent singularity.

A fundamental problem concerning time-dependent singularities is to find a condition for the removability of the singularities. It was shown that a solution u of the homogeneous linear heat equation in $\bigcup_{t\in I}((\Omega \setminus \{\xi(t)\}) \times \{t\})$ is smoothly extended to $\Omega \times I$ if and only if $u(x,t) = o(|x-\xi(t)|^{2-N})$ locally uniformly for $t \in I$ as $|x-\xi(t)| \to 0$ (see [15,16,28]). In [14], this result was verified also for nonlinear equations including (1.1), provided that $p < p_{sg} := N/(N-2)$ and $\xi(t) \equiv 0$. We remark that the exponent p_{sg} is a threshold for the existence of an explicit singular steady state of (1.1). Indeed, one can easily check that the function $L|x|^{-2/(p-1)}$ is a stationary solution of (1.1) with $\xi(t) \equiv 0$ if $p > p_{sg}$, where the constant L is given by

$$L := \left\{ \frac{2}{p-1} \left(N - 2 - \frac{2}{p-1} \right) \right\}^{\frac{1}{p-1}}. \tag{1.2}$$

This particularly shows that the condition $u(x,t) = o(|x-\xi(t)|^{2-N})$ $(|x-\xi(t)| \to 0)$ is not sufficient for removability if $p > p_{sg}$ and $\xi(t) \equiv 0$.

Another problem is the existence of solutions with a time-dependent singularity. In the case where $p < p_{sg}$, $\xi(t) \equiv 0$, $I = (0,\infty)$ and Ω is bounded, the existence of a positive solution u of (1.1) satisfying $u|_{\partial\Omega\times I} = 0$ and $\lim_{x\to 0} u(x,t) = +\infty$ $(t \in I)$ was shown in [29]. The case where ξ is nonconstant was first considered in [25]. It was proved that if $p_{sg} < p < p_* := (N + 2\sqrt{N-1})/(N - 4 + 2\sqrt{N-1})$, $\Omega = \mathbb{R}^N$, $I = (0,T)$ and $T > 0$ is small enough, then there is a positive solution u of (1.1) such that

$$u(x,t) = (L + o(1))|x - \xi(t)|^{-\frac{2}{p-1}} \text{ for every } t \in I \text{ as } x \to \xi(t). \tag{1.3}$$

In [26], the existence of a solution satisfying (1.3) for $I = (0,\infty)$ was studied. Recently, a solution with a time-dependent singularity was constructed in the Navier–Stokes system (see [20]).

The problem in this paper is to find all possible behaviors of solutions of (1.1) at the time-dependent singularity. To approach this problem, we first prove that every nonnegative classical solution u of (1.1) is extended as a distributional solution of $\partial_t u - \Delta u = u^p + \Lambda$ in $\mathcal{D}'(\Omega \times I)$, where Λ is some measure whose support is contained in $\{(\xi(t),t); t \in I\}$ (see Theorem 2.1). We then observe that the behavior at the time-dependent singularity is governed by the measure Λ, provided that $p < p_{sg}$. In fact, there is a function $a \in L^1_{\mathrm{loc}}(I)$ determined by Λ such that $u(x,t) = (a(t) + o(1))|x-\xi(t)|^{2-N}$ for almost every $t \in I$ as $x \to \xi(t)$ (see Theorem 2.3). In addition, we also study the properties of the measure Λ (see Theorem 2.2).

This paper is organized as follows. In Section 2, we state main results and remark relations between the results and previous works. In Section 3, we examine a solution of the linear heat equation having a time-dependent singularity. We will utilize this solution not only to analyze the behavior of solutions of (1.1), but also to construct suitable test functions used in showing the existence of the measure Λ. Sections 4–6 are devoted to proving the results.