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Yamabe type equations with a sign-changing nonlinearity, and the prescribed curvature problem

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Abstract

In this paper, we investigate the prescribed scalar curvature problem on a non-compact Riemannian manifold $(M, \langle\, , \,\rangle)$, namely the existence of a conformal deformation of the metric $\langle\, , \,\rangle$ realizing a given function $\widetilde{s}(x)$ as its scalar curvature. In particular, the work focuses on the case when $\widetilde{s}(x)$ changes sign. Our main achievement are two new existence results requiring minimal assumptions on the underlying manifold, and ensuring a control on the stretching factor of the conformal deformation in such a way that the conformally deformed metric be bi-Lipschitz equivalent to the original one. The topological–geometrical requirements we need are all encoded in the spectral properties of the standard and conformal Laplacians of M. Our techniques can be extended to investigate the existence of entire positive solutions of quasilinear equations of the type

$$\Delta_p u + a(x)u^{p-1} - b(x)u^\sigma = 0$$

where Δ_p is the p-Laplacian, $\sigma > p-1 > 0$, $a, b \in L^\infty_{\mathrm{loc}}(M)$ and b changes sign, and in the process of collecting the material for the proof of our theorems, we have the opportunity to give some new insight on the subcriticality theory for the Schrödinger type operator

$$Q'_V \,:\, \varphi \longmapsto -\Delta_p\varphi - a(x)|\varphi|^{p-2}\varphi.$$

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In particular, we prove sharp Hardy-type inequalities in some geometrically relevant cases, notably for minimal submanifolds of the hyperbolic space.

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Keywords: Yamabe equation; Schrödinger operator; Subcriticality; p-Laplacian; Spectrum; Prescribed curvature

Contents

1. Introduction, I: existence for the generalized Yamabe problem

Generalizations of the classical Yamabe problem on a Riemannian manifold have been the focus of an active area of research over the past 30 years. Among these, the prescribed scalar curvature problem over non-compact manifolds appears to be challenging: briefly, given a non-compact Riemannian manifold $(M^m, \langle\, , \rangle)$ with scalar curvature $s(x)$ and a smooth function $\widetilde{s} \in C^\infty(M)$, the problem asks under which conditions there exists a conformal deformation of $\langle\, , \rangle$,

$$\widetilde{\langle\, , \rangle} = \varphi^2 \langle\, , \rangle, \qquad 0 < \varphi \in C^\infty(M), \tag{1.1}$$

realizing $\widetilde{s}(x)$ as its scalar curvature. When the dimension m of M is at least 3, writing $\varphi = u^{\frac{2}{m-2}}$, the problem becomes equivalent to determining a positive solution $u \in C^\infty(M)$ of the Yamabe equation

$$\Delta u - \frac{s(x)}{c_m} u + \frac{\widetilde{s}(x)}{c_m} u^{\frac{m+2}{m-2}} = 0, \qquad c_m = \frac{4(m-1)}{m-2}. \tag{1.2}$$

Here, Δ is the Laplace–Beltrami operator of the background metric $\langle\, , \rangle$. For $m = 2$, setting $\varphi = e^u$ one substitutes (1.2) with