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Local well-posedness for the fifth-order KdV equations on $\mathbb{T}$

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Abstract

This paper is a continuation of the paper *Low regularity Cauchy problem for the fifth-order modified KdV equations on* $\mathbb{T}$ [7]. In this paper, we consider the fifth-order equation in the Korteweg–de Vries (KdV) hierarchy as following:

$$\begin{cases} \partial_t u - \partial_x^5 u - 30u^2\partial_x u + 20\partial_x u\partial_x^2 u + 10u\partial_x^3 u = 0, & (t,x) \in \mathbb{R}\times\mathbb{T}, \\ u(0,x) = u_0(x) \in H^s(\mathbb{T}). \end{cases}$$

We prove the local well-posedness of the fifth-order KdV equation for low regularity Sobolev initial data via the energy method. This paper follows almost same idea and argument as in [7]. Precisely, we use some conservation laws of the KdV Hamiltonians to observe the direction which the nonlinear solution evolves to. Besides, it is essential to use the short time $X^{s,b}$ spaces to control the nonlinear terms due to *high* $\times$ *low* $\Rightarrow$ *high* interaction component in the non-resonant nonlinear term. We also use the localized version of the modified energy in order to obtain the energy estimate.

As an immediate result from a conservation law in the scaling sub-critical problem, we have the global well-posedness result in the energy space H^2.

MSC: 35Q53; 37K10

Keywords: The fifth-order KdV equation; Local well-posedness; Energy method; Complete integrability; $X^{s,b}$ space; Modified energy

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1. Introduction

The periodic Korteweg–de Vries (KdV) equation

$$\partial_t u + \partial_x^3 u + 6u\partial_x u = 0$$

is completely integrable in the sense that the equation admits *Lax pair* representations. Thanks to the inverse spectral method, it is well known that the KdV equation has a global smooth solution for any smooth initial data. Moreover, from the fact that the integrable Hamiltonian systems have the bi-Hamiltonian structure, there are infinitely many equations and corresponding Hamiltonians (so-called KdV hierarchy), and every equation in the hierarchy enjoys all conservation laws. The following are few conservation laws in the hierarchy:

$$M[u] := \int \frac{1}{2}u, \qquad E[u] := \int \frac{1}{2}u^2 \tag{1.1}$$

In this paper, we consider the following integrable fifth-order KdV equation:

$$\begin{cases} \partial_t u - \partial_x^5 u - 30u^2\partial_x u + 20\partial_x u\partial_x^2 u + 10u\partial_x^3 u = 0, & (t,x) \in \mathbb{R}\times\mathbb{T}, \\ u(0,x) = u_0(x) \in H^s(\mathbb{T}) \end{cases} \tag{1.2}$$

where $\mathbb{T} = [0, 2\pi]$. Even if (1.2) and the other equations in the hierarchy have the integrable structure, it is still required the analytic theory of nonlinear dispersive equations to solve the low regularity Cauchy problem. In fact, in previous studies on the low regularity well-posedness problem for nonlinear dispersive equations (especially, under the non-periodic setting), the integrable structures were ignored. This work is a continuation of the paper *Low regularity Cauchy problem for the fifth-order modified KdV equations on* $\mathbb{T}$ [7] to show that, in the periodic setting, the complete integrability is partly needed to study on the low regularity well-posedness problem.[1]

Generalizing coefficients in the nonlinear terms may break the integrable structure. The following equation generalizes (1.2) to non-integrable case:

$$\begin{cases} \partial_t u - \partial_x^5 u + a_1 u^2\partial_x u + a_2\partial_x u\partial_x^2 u + a_3 u\partial_x^3 u = 0, & (t,x) \in \mathbb{R}\times\mathbb{T}, \\ u(0,x) = u_0(x) \in H^s(\mathbb{T}), \end{cases} \tag{1.3}$$

where a_i's, $i = 1, 2, 3$, are real constants. For studying (1.3), we can rely no longer on the property of the complete integrability.

Meanwhile, once one observes the Fourier coefficients of both (1.2) and (1.3) (see (2.1) below), one can find, in the nonlinear interactions, some resonant terms such as

$$\int_{\mathbb{T}} u(t,x)\,dx \cdot \partial_x^3 u, \quad \|u(t)\|_{L^2}^2 \partial_x u,$$

[1] In fact, even if the integrability is neglected completely, the same result can be obtained for the integrable and also non-integrable equations. See Theorem 1.3.