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Existence of traveling waves of auto-catalytic systems with decay

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Abstract

This article establishes the existence of traveling waves of a class of reaction–diffusion systems which model the pre-mixed isothermal autocatalytic chemical reaction of order m ($m > 1$) between two chemical species, a reactant and an auto-catalyst, and a linear decay. Moreover, our result shows that the set of speed is contained in a bounded interval for any fixed initial value at $x = -\infty$. This is in strong contrast to either the reaction–diffusion systems of autocatalytic chemical reaction of the order m without decay, or to the systems which have the same order of decay, which were shown by various authors (e.g. [8,17,13,26]) that the set of traveling wave speeds contains $[c_*, \infty)$ for some $c_* > 0$.

The same systems also appear in a mathematical model of microbial growth and competition in a flow reactor; see [2,24].

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1. Introduction

In this paper we consider a reaction–diffusion system

$$u_t = d_1 u_{xx} - k_1 u v^m, \qquad v_t = d_2 v_{xx} + k_1 u v^m - k_2 v, \tag{1.1}$$

where $d_1 > 0$, $d_2 > 0$, $m > 1$ and $k_1 > 0$, $k_2 > 0$ are constants. It models an auto-catalytic chemical reaction with a decay step:

$$A + mB \longrightarrow (m+1)B \quad \text{with rate } k_1[A][B]^m, \qquad B \longrightarrow C \quad \text{with rate } k_2[B],$$

where C is an inert product. It also appears in a mathematical model of microbial growth and competition in a flow reactor; see [2] and [24].

By scaling, we can without loss of generality assume that $d_1 = 1$, $d_2 = d$, $k_1 = 1$, and $k_2 = 1$.

It is easy to verify that all equilibrium points of (1.1) are in the form $(a, 0)$ with $a \in R$. Hence, any traveling wave $u(x,t) = u(x - ct)$, $v(x,t) = v(x - ct)$, with $c > 0$ as the speed, must link one equilibrium point $(u_0, 0)$ at $x = -\infty$ to another one $(u_1, 0)$ at $x = \infty$ with $u_1 > u_0 > 0$, and the equilibrium points are saddle in nature. Thus, we consider traveling wave problem

$$\begin{cases} u'' + cu' = uv^m, & u' > 0 \quad \text{in } \mathbb{R}, \\ dv'' + cv' = v - uv^m, & v > 0 \quad \text{in } \mathbb{R}, \\ u(-\infty) = u_0, \quad v(-\infty) = 0, & v(\infty) = 0, \quad u(\infty) < \infty. \end{cases} \tag{1.2}$$

The advantage of (1.1), from a modeling point of view, over the system

$$u_t = d_1 u_{xx} - k_1 u v^m, \qquad v_t = d_2 v_{xx} + k_1 u v^m,$$

which models auto-catalytic chemical reaction without a decay step, is that it was demonstrated in [5] by asymptotic analysis and numerical computation, and rigorously proved for $m = 1$ in [9] that any small amount of B introduced locally with uniform initial distribution of A can generate traveling wave. This feature, which should hold true for general m order reaction, seems to be contradictory to the fact that in the relevant experimental result of chemical reactions, the initiation of traveling wave calls for sufficient amount of B to be added [27]. To overcome this deficiency in modeling for lacking of threshold phenomenon, Gray [12] made the observation that the auto-catalyst B cannot be stable indefinitely and should be used to produce other chemicals. In particular, it was suggested in [19,20] that B decays to an inert product C at a rate of order n,

$$B \longrightarrow C \qquad \text{with rate } kv^n.$$

The resulting PDE system is

$$\begin{cases} u_t = u_{xx} - uv^m \\ v_t = dv_{xx} + uv^m - kv^n, \end{cases}$$

where $m, n \geq 1$ and $k > 0$ is a rate constant, after a simple scaling.

If $m = n$, the above system takes the form: