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Large time behavior for the fast diffusion equation with critical absorption

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Abstract

We study the large time behavior of nonnegative solutions to the Cauchy problem for a fast diffusion equation with critical zero order absorption

$$\partial_t u - \Delta u^m + u^q = 0 \qquad \text{in } (0,\infty) \times \mathbb{R}^N ,$$

with $m_c := (N-2)_+/N < m < 1$ and $q = m + 2/N$. Given an initial condition u_0 decaying arbitrarily fast at infinity, we show that the asymptotic behavior of the corresponding solution u is given by a Barenblatt profile with a logarithmic scaling, thereby extending a previous result requiring a specific algebraic lower bound on u_0. A by-product of our analysis is the derivation of sharp gradient estimates and a universal lower bound, which have their own interest and hold true for general exponents $q > 1$.

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Keywords: Large time behavior; Fast diffusion; Critical absorption; Gradient estimates; Lower bound

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1. Introduction and main results

In this paper, we deal with the large time behavior of a fast diffusion equation with absorption, in a special case when the exponent of the absorption term is critical. More precisely, we consider the following Cauchy problem

$$\partial_t u - \Delta u^m + u^q = 0 \quad \text{in } (0,\infty) \times \mathbb{R}^N, \tag{1.1}$$

with initial condition

$$u(0,x) = u_0(x), \qquad x \in \mathbb{R}^N, \tag{1.2}$$

where $N \geq 1$,

$$u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \quad u_0 \geq 0,\ u_0 \not\equiv 0, \tag{1.3}$$

and the parameters m and q satisfy

$$m_c := \frac{(N-2)_+}{N} < m < 1, \quad q = q_* := m + \frac{2}{N}. \tag{1.4}$$

Degenerate and singular parabolic equations with absorption such as (1.1) have been the subject of intensive research during the last decades. In (1.1), the main feature is the competition between the diffusion Δu^m and the absorption $-u^q$ which turns out to depend heavily on the exponents $m > 0$ and $q > 0$. More precisely, a critical exponent $q_* = m + 2/N$ has been uncovered which separates different dynamics and the large time behavior for non-critical exponents $q \neq q_*$ is now well understood. Indeed, for the semilinear case $m = 1$ and the slow diffusion case $m > 1$, it has been shown that, when $q > q_*$, the effect of the absorption is negligible, and the large time behavior is given by the diffusion alone, leading to either Gaussian or Barenblatt profiles [9,14,15,18,19,21].

A more interesting case turns out to be the intermediate range of the absorption exponent $q \in (m, q_*)$, where the competition of the two effects is balanced. For $m \geq 1$, the study of this range has led to the discovery of some special self-similar solutions called *very singular solutions* which play an important role in the description of the large time behavior, see [3,6,9,15,19–21,26] for instance. This was an important improvement, as the existence of very singular solutions has been later established for many other different equations.

The study of the fast diffusion case $0 < m < 1$ was performed later, but restricted to the range of exponents $m_c < m < 1$, as the singular phenomenon of finite time extinction occurs when $m \in (0, m_c)$. When $m \in (m_c, 1)$, the asymptotic behavior has been also identified for any $q \neq q_*$, $q > 1$, and again very singular solutions play an important role [25,27,28]. Later, also the extinction case when q ranges in $(0, 1)$ has been studied [7,8], although there are still many open problems in these ranges, as most of the results are valid only in dimension $N = 1$.

In this paper we focus on the critical absorption exponent $q = q_*$ which is the limiting case above which the effect of the absorption term is negligible in the large time dynamics. That the diffusion is almost governing the asymptotic behavior is revealed by the fact that the asymptotic profile is given by the diffusion, but the scaling is modified as a result of the influence of the absorption term and additional logarithmic factors come into play. More precisely, the solutions