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Second order evolution equations which describe pseudospherical surfaces

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Abstract

Second order evolution differential equations that describe pseudospherical surfaces are considered. These equations are equivalent to the structure equations of a metric with Gaussian curvature $K = -1$, and can be seen as the compatibility condition of an associated $\mathfrak{sl}(2, \mathbb{R})$-valued linear problem, also referred to as a zero curvature representation. Under the assumption that the linear problem is defined by 1-forms $\omega_i = f_{i1}dx + f_{i2}dt$, $i = 1, 2, 3$, with f_{ij} depending on (x, t, z, z_1, z_2) and such that $f_{21} = \eta$, $\eta \in \mathbb{R}$, we give a complete and explicit classification of equations of the form $z_t = A(x, t, z)\, z_2 + B(x, t, z, z_1)$. According to the classification, these equations are subdivided in three main classes (referred to as Types I–III) together with the corresponding linear problems. Explicit examples of differential equations of each type are determined by choosing certain arbitrary differentiable functions. Svinolupov–Sokolov equations admitting higher weakly nonlinear symmetries, Boltzmann equation and reaction–diffusion equations like Murray equation are some known examples of such equations. Other explicit examples are presented, as well.

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1. Introduction

Differential equations which describe pseudospherical surfaces arise ubiquitously as suitable models in the description of nonlinear physical phenomena as well as in many problems of pure and applied mathematics. Geometrically these equations are characterized by the fact that their generic solutions provide metrics on open subsets of $\mathbb{R}^2$, with Gaussian curvature $K = -1$. The first well known example of such an equation is the sine-Gordon equation $z_{xt} = \sin(z)$. This example was discovered by Edmond Bour [2], who realized that in terms of Darboux asymptotic coordinates the Gauss–Mainardi–Codazzi equations for pseudospherical surfaces contained in $\mathbb{R}^3$ reduce to the sine-Gordon equation. Then, the discovery of Bäcklund transformations first, and later the construction by Bianchi of the superposition formula for solutions of this equation, focused even more attention on the sine-Gordon equation, that in the end it turned out to be an important model in the description of several nonlinear phenomena (see for example [23, 25,39]). However, it was after the early observation [37] that "all the soliton equations in $1 + 1$ dimensions that can be solved by the AKNS 2×2 inverse scattering method (for example, the sine-Gordon, KdV or modified KdV equations) ... describe pseudospherical surfaces", that the general study of these equations was initiated. In particular, it was with the fundamental paper [15] by S.S. Chern and K. Tenenblat that initiated a systematic study of these equations. The results of this study, together with the considerable effort addressed over the past few decades to the possible applications of inverse scattering method, gave a significant contribution to the discovery of new integrable equations. For instance, Belinski–Zakharov system in General Relativity [7], the nonlinear Schrödinger type systems [14,17,19], the Rabelo's cubic equation [5,28, 29,36], the Camassa–Holm, Degasperis–Procesi, Kaup–Kupershmidt and Sawada–Kotera equations [8,9,32–35] are some important examples of equations describing pseudospherical surfaces and integrable by inverse scattering method. All these facts prove the relevance of these equations and justify our general interest in their study and classification.

A differential equation for a real function $z(x, t)$ is said to describe pseudospherical surfaces, or to be a PS equation, if it is equivalent to the structure equations $d\omega_1 = \omega_3 \wedge \omega_2$, $d\omega_2 = \omega_1 \wedge \omega_3$, $d\omega_3 = \omega_1 \wedge \omega_2$ of a 2-dimensional Riemannian manifold whose Gaussian curvature $K = -1$, with 1-forms $\omega_i = f_{i1}dx + f_{i2}dt$ satisfying the non-degeneracy condition $\omega_1 \wedge \omega_2 \neq 0$ and such that f_{ij} are smooth functions of x, t, z and derivatives of z with respect to x and t.

In [15] Chern and Tenenblat obtained characterization results for evolution equations of the form $z_t = F(z, z_1, \ldots, z_k)$ (from now on we denote $z_i = \partial^i z/\partial x^i$), under the assumptions that $f_{ij} = f_{ij}(z, z_1, \ldots, z_k)$ and $f_{21} = \eta$, where η is a parameter. In the same paper the authors also considered a similar problem for equations of the form $z_{1,t} = F(z, z_1, \ldots, z_k)$. A noteworthy result of this study was an effective method for the explicit determination of entire new classes of differential equations that describe pseudospherical surfaces. Motivated by the results of [15], in a series of subsequent papers [22,28–30], the same method was systematically implemented and new classes of pseudospherical equations were identified still with the basic assumption that $f_{21} = \eta$. Then in [13] the authors showed how the geometric properties of pseudospherical surfaces may provide infinite number of conservation laws when the functions f_{ij} are analytic functions of the parameter η. This parameter however is important not only for the existence of infinite number of conservation laws, but is also related to the existence of Bäcklund transformations and is crucial in the application of inverse scattering method, as shown in [5,15].

In 1995, Kamran and Tenenblat [24] generalized the results of [15] by giving a complete characterization of evolution equations of type $z_t = F(z, z_1, \ldots, z_k)$ which describe pseudospherical surfaces, in terms of necessary and sufficient conditions that have to be satisfied by F and the