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Integrable deformations of Rössler and Lorenz systems from Poisson–Lie groups

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Abstract

A method to construct integrable deformations of Hamiltonian systems of ODEs endowed with Lie–Poisson symmetries is proposed by considering Poisson–Lie groups as deformations of Lie–Poisson (co)algebras. Moreover, the underlying Lie–Poisson symmetry of the initial system of ODEs is used to construct integrable coupled systems, whose integrable deformations can be obtained through the construction of the appropriate Poisson–Lie groups that deform the initial symmetry. The approach is applied in order to construct integrable deformations of both uncoupled and coupled versions of certain integrable types of Rössler and Lorenz systems. It is worth stressing that such deformations are of non-polynomial type since they are obtained through an exponentiation process that gives rise to the Poisson–Lie group from its infinitesimal Lie bialgebra structure. The full deformation procedure is essentially algorithmic and can be computerized to a large extent.

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1. Introduction

The integrability problem for systems of first order ODEs is indeed a relevant issue in the theory of dynamical systems (see, for instance, [1–5] and references therein), and the same question for coupled systems of ODEs could be also meaningful from the viewpoint of applications (for instance, regarding synchronization problems [6]). In this context, the aim of this paper is two fold: on one hand, to present a novel approach to the explicit construction of Liouville-integrable coupled systems of ODEs endowed with Lie–Poisson symmetries and, on the other hand, to show how integrable deformations of the previous systems can be systematically obtained.

The construction here presented establishes a (to the best of our knowledge) novel connection between Poisson–Lie groups and integrable deformations of dynamical systems. Such a link is based on the well-known result by Drinfel'd that establishes the one-to-one correspondence between Poisson–Lie groups and Lie bialgebras [7], as well as on the general construction of integrable Hamiltonian systems from Poisson coalgebras that was introduced in [8–10], thus giving rise to the so-called "coalgebra symmetry approach" to finite dimensional integrable systems (see [11,12] for applications of this method to Hénon–Heiles and Lotka–Volterra systems, respectively, and [13] for the coalgebra-based construction of nonlinear superposition rules arising in nonautonomous Lie–Hamilton systems [14]).

In particular, we will apply the formalism here introduced in order to obtain new integrable deformations of certain Lorenz [15] and Rössler [16] systems, as well as of some coupled versions of them. Nevertheless, we stress that the method presented in the paper is completely general and could be applied to any other interesting Hamiltonian systems of ODEs endowed with a Lie–Poisson symmetry.

The basics of the Poisson coalgebra approach to integrability will be summarized in the next Section. In short, we will consider a finite-dimensional integrable Hamiltonian system of ODEs that is defined through a Lie–Poisson algebra $(\mathcal{F}(g^*), \{\,,\})$, where $\mathcal{F}(g^*)$ is the algebra of smooth functions on g^* and the Lie–Poisson bracket is $\{\,,\}$. A set of functions in involution will be given by a single Hamiltonian function $\mathcal{H} \in \mathcal{F}(g^*)$ and a number of Casimir functions $\mathcal{C}_k \in \mathcal{F}(g^*)$, $k = 1, \ldots, r$. Then, since Lie–Poisson algebras are endowed with a (primitive or non-deformed) coalgebra structure, the coalgebra symmetry approach will straightforwardly provide, by following [8], a coupled integrable generalization of the system. This approach will be illustrated in Section 3 through the explicit construction of an integrable coupled Rössler system.

A detailed technical presentation of the construction of Poisson–Lie groups as deformations of Lie bialgebras will be given in Section 4. If we denote by g the abstract Lie algebra associated with the Lie–Poisson coalgebra $(\mathcal{F}(g^*), \{\,,\})$, we will assume that we are able to find a cocommutator map δ compatible with g, so that (g, δ) defines a Lie bialgebra structure [17,18]. Then, it is well-known that the dual of the cocommutator δ will define a second Lie algebra structure (that we will denote by d), and the dual of the structure tensor for g will provide a compatible cocommutator map γ for d, in such a way that (d, γ) is just the dual Lie bialgebra to (g, δ). Once a given $\delta^* \simeq d$ is identified, the (connected component containing the identity) of the Lie group D with Lie algebra d can be constructed by exponentiation, and the unique Poisson–Lie structure having g^* as its linearization at the identity can be computed, thus providing the deformed Poisson structure $\{\,,\}_\eta$ we were looking for. Simultaneously, the group law for D will give us the explicit deformed coproduct map Δ_η on the algebra of smooth functions $\mathcal{F}(D)$. Moreover, the deformation nature of this construction arises in a more transparent way if we consider the one-parametric cocommutator $\gamma_\eta = \eta\,\gamma$, since the parameter η will give us information concerning the orders in the deformation process that arises from the 'exponentiation' of the tangent Lie