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J. Differential Equations 260 (2016) 5678–5725

Journal of Differential Equations

www.elsevier.com/locate/jde

A new div–curl result. Applications to the homogenization of elliptic systems and to the weak continuity of the Jacobian

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Received 16 March 2015; revised 4 December 2015

Available online 4 January 2016

Abstract

In this paper a new div–curl result is established in an open set Ω of $\mathbb{R}^N$, $N \geq 2$, for the product $\sigma_n \cdot \eta_n$ of two sequences of vector-valued functions σ_n, η_n such that σ_n is bounded in $L^p(\Omega)^N$, η_n is bounded in $L^q(\Omega)^N$, with $1/p + 1/q = 1 + 1/(N-1)$, and such that $\operatorname{div} \sigma_n$, $\operatorname{curl} \eta_n$ are compact in suitable spaces. The new assumption is that the product converges weakly in $W^{-1,1}(\Omega)$. The approach is also new in the topic, and is based on a compactness result for bounded sequences in $W^{1,q}(\Omega)$ through a suitable selection of annuli on which the gradients are not too high, in the spirit of [26,32] and using the imbedding of $W^{1,q}$ into $L^{p'}$ for the unit sphere of $\mathbb{R}^N$. The div–curl result is applied to the homogenization of equi-coercive systems whose coefficients are equi-bounded in $L^\rho(\Omega)$ for some $\rho > \frac{N-1}{2}$ if $N > 2$, or in $L^1(\Omega)$ if $N = 2$. It also allows us to prove a weak continuity result for the Jacobian for bounded sequences in $W^{1,N-1}(\Omega)$ satisfying an alternative assumption to the L^∞-strong estimate of [8]. Two examples show the sharpness of the results.

MSC: 35B27; 74Q15

Keywords: Div–curl; Homogenization; Elliptic systems; Non-equi-bounded coefficients; Γ-convergence; Jacobian

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http://dx.doi.org/10.1016/j.jde.2015.12.029
0022-0396/

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1. Introduction

In the early 1970s Murat and Tartar noticed that for any sequence σ_n weakly converging to σ in $L^p_{\rm loc}(\mathbb{R}^N)$, $N\geq 2$ and $p\in(1,\infty)$, and any sequence u_n converging weakly to u in $W^{1,p'}_{\rm loc}(\mathbb{R}^N)$ such that $\operatorname{div}\sigma_n$ converges strongly in $W^{-1,p}_{\rm loc}(\mathbb{R}^N)$, a simple integration by parts leads to the convergence

$$\sigma_n\cdot\nabla u_n \rightharpoonup \sigma\cdot\nabla u \quad \text{in } \mathscr{D}'(\mathbb{R}^N). \tag{1.1}$$

They extended this remark to the more general case where ∇u_n is replaced by any sequence η_n such that curl η_n is compact in $W^{-1,p'}_{\rm loc}(\mathbb{R}^N)$ (see [37]). The successful compensated compactness theory was born with a fruitful application to homogenization theory [36]. Actually, the elementary div–curl (1.1) contains hidden information. Indeed, Coifman et al. proved that if $\operatorname{div}\sigma$ is in $W^{-1,s}_{\rm loc}(\mathbb{R}^N)$ with $s>p$, then $\sigma\cdot\nabla u$ belongs to the Hardy space $\mathscr{H}^1_{\rm loc}(\mathbb{R}^N)$. More recently, Conti et al. [21] obtained a new div–curl result relaxing the compensation conditions on $\operatorname{div}\sigma_n$ and curl η_n to the space $W^{-1,1}_{\rm loc}(\mathbb{R}^N)$, but assuming that the sequence $\sigma_n\cdot\eta_n$ is equi-integrable. On the other hand, in the spirit of [36,37] and using an appropriate Hodge decomposition of vector-valued fields, it was proved in [15] that, given a bounded open set Ω of $\mathbb{R}^N$, $N\geq 2$, if $p,q\in[1,\infty)$ satisfy

$$p,q\geq 1,\qquad \frac{1}{p}+\frac{1}{q}\leq 1+\frac{1}{N}, \tag{1.2}$$

and if σ_n, η_n satisfy the convergences

$$\sigma_n\rightharpoonup\sigma\ \begin{cases} L^p(\Omega)^N, & \text{if } p>1\\ \mathscr{M}(\Omega)^N *, & \text{if } p=1,\end{cases}\qquad \eta_n\rightharpoonup\eta\ \begin{cases} L^q(\Omega)^N, & \text{if } q>1\\ \mathscr{M}(\Omega)^N *, & \text{if } q=1,\end{cases} \tag{1.3}$$