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The 16th Hilbert problem restricted to circular algebraic limit cycles

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Abstract

We prove the following two results. First every planar polynomial vector field of degree S with S invariant circles is Darboux integrable without limit cycles. Second a planar polynomial vector field of degree S admits at most $S-1$ invariant circles which are algebraic limit cycles. In particular we solve the 16th Hilbert problem restricted to algebraic limit cycles given by circles, because a planar polynomial vector field of degree S has at most $S-1$ algebraic limit cycles given by circles, and this number is reached.

MSC: 34C07

Keywords: Algebraic limit circles; Planar polynomial differential system; Polynomial vector fields; Invariant algebraic circles; Darboux integrability; 16th Hilbert's problem

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1. Introduction and statement of the main results

Let $\mathbb{R}[x, y]$ be the ring of all real polynomials in the variables x and y. Assume that $P, Q \in \mathbb{R}[x, y]$ such that P and Q are coprime in $\mathbb{R}[x, y]$. Consider the set Σ of all planar real polynomial vector fields

$$\mathcal{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial x},$$

associated to the differential polynomial systems

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y),$$

of degree $m = \max\{\deg P, \deg Q\}$, here the dot denotes derivative respect to the time t.

Let U be an open and dense set in $\mathbb{R}^2$. We say that a non-constant C^1 function $H : U \to \mathbb{R}$ is a *first integral* of the polynomial vector field $\mathcal{X}$ on U, if $H(x(t), y(t))$ is constant for all values of t for which the solution $(x(t), y(t))$ of $\mathcal{X}$ is defined on U. Clearly H is a first integral of $\mathcal{X}$ on U if and only if $\mathcal{X}H = 0$ on U.

Let $g = g(x, y) \in \mathbb{R}[x, y]$. Then Let $g = 0$ be an *invariant algebraic curve* of $\mathcal{X}$ if

$$\mathcal{X}g = P\frac{\partial g}{\partial x} + Q\frac{\partial g}{\partial y} = Kg,$$

where $K = K(x, y)$ is a polynomial of degree at most $m - 1$, which is called the *cofactor* of $g = 0$. If the polynomial g is irreducible in $\mathbb{R}[x, y]$, then we say that the invariant algebraic curve $g = 0$ is *irreducible* and that its *degree* is the degree of the polynomial g.

We recall that a *limit cycle* of a polynomial vector field $\mathcal{X}$ is an isolated periodic orbit in the set of all periodic orbits of $\mathcal{X}$. An *algebraic limit cycle of degree n* of $\mathcal{X}$ is an oval of an irreducible invariant algebraic curve $g = 0$ of degree n, which is a limit cycle of $\mathcal{X}$.

Hilbert in [5] asked: *Is there an upper bound for the maximum number of limit cycles of any polynomial vector field with a given degree*? This is a version of the second half part of the *Hilbert's 16-th problem*. This problem remains open, see for more information [3,4,6,7,14].

A simpler version of the second part of the 16-th Hilbert's problem restricted to algebraic limit cycles can be stated as follows: Consider the set Σ'_m of all real polynomial vector fields $\mathcal{X}$ of degree m having real invariant algebraic curves. *Is there an upper bound on the maximum number of algebraic limit cycles of any polynomial vector field of* Σ'_m? (see [8,9]).

There is the following conjecture (see [9]) about the maximum number of algebraic limit cycles of polynomial vector fields with a given degree.

Conjecture. *The maximum number of algebraic limit cycles that a polynomial vector field of degree* $m \geq 2$ *can have is* $1 + (m-1)(m-2)/2$.

This conjecture has been proved when the invariant algebraic curves of the polynomial vector fields satisfy some generic properties see [9], see also [8,10,15].

Let $f_i, g_j, h_j \in \mathbb{R}[x, y]$ for $i = 1, \ldots, p$ and $j = 1, \ldots, q$. Then the (multi-valued) function

$$|f_1|^{\lambda_1} \cdots |f_p|^{\lambda_p} e^{\mu_1 g_1/h_1} \cdots e^{\mu_q g_q/h_q}$$

with $\lambda_i, \mu_j \in \mathbb{C}$ is called a *(generalized) Darboux function.*