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Convergence rates in homogenization of Stokes systems

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Abstract

This paper studies the convergence rates in L^2 and H^1 of Dirichlet problems for Stokes systems with rapidly oscillating periodic coefficients, without any regularity assumptions on the coefficients.

Keywords: Convergence rates; Stokes systems; Homogenization

1. Introduction and main results

The purpose of this paper is to study the convergence rates of Dirichlet problems for Stokes systems with rapidly oscillating periodic coefficients. More precisely, we consider the following Dirichlet problem for Stokes systems associated with matrix A,

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) + \nabla p_\varepsilon = F & \text{in } \Omega, \\ \operatorname{div} u_\varepsilon = g & \text{in } \Omega, \\ u_\varepsilon = f & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

with the compatibility condition

$$\int_\Omega g - \int_{\partial\Omega} f \cdot n = 0, \tag{1.2}$$

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where n denotes the outward unit normal to $\partial\Omega$ and $\Omega \subset \mathbb{R}^d$ is a bounded domain. We note that the Dirichlet problem (1.1) is used in the modeling of flows in porous media. Here $\varepsilon > 0$ is a small parameter and the operator $\mathcal{L}_\varepsilon$ is defined by

$$\mathcal{L}_\varepsilon = -\mathrm{div}(A(x/\varepsilon)\nabla) = -\frac{\partial}{\partial x_i}\left[a_{ij}^{\alpha\beta}\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_j}\right] \tag{1.3}$$

with $1 \le i, j, \alpha, \beta \le d$ (the summation convention is used throughout). We will assume that the coefficient matrix $A(y) = (a_{ij}^{\alpha\beta}(y))$ is real, bounded measurable, and satisfies the ellipticity condition:

$$\mu|\xi|^2 \le a_{ij}^{\alpha\beta}(y)\xi_i^\alpha \xi_j^\beta \le \frac{1}{\mu}|\xi|^2 \qquad \text{for } y \in \mathbb{R}^d \text{ and } \xi = (\xi_i^\alpha) \in \mathbb{R}^{d\times d}, \tag{1.4}$$

where $\mu > 0$. We also assume that $A(y)$ satisfies the periodicity condition,

$$A(y+z) = A(y) \qquad \text{for } y \in \mathbb{R}^d \text{ and } z \in \mathbb{Z}^d. \tag{1.5}$$

No symmetry condition on $A(y)$ is needed. A function satisfying (1.5) will be called 1-periodic.

By the homogenization theory of Stokes systems (see [2,6]), under suitable conditions on F, f and g, it is known that

$$u_\varepsilon \rightharpoonup u_0 \quad \text{weakly in } H^1(\Omega; \mathbb{R}^d) \quad \text{and} \quad p_\varepsilon - \fint_\Omega p_\varepsilon \rightharpoonup p_0 - \fint_\Omega p_0 \quad \text{weakly in } L^2(\Omega),$$

where $(u_0, p_0) \in H^1(\Omega; \mathbb{R}^d) \times L^2(\Omega)$ is the weak solution of the homogenized problem with constant coefficients,

$$\begin{cases} \mathcal{L}_0(u_0) + \nabla p_0 = F & \text{in } \Omega, \\ \mathrm{div}\, u_0 = g & \text{in } \Omega, \\ u_0 = f & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

The primary purpose of this paper is to investigate the rate of convergence of $\|u_\varepsilon - u_0\|_{L^2(\Omega)}$, as $\varepsilon \to 0$. The following is the main result of the paper.

Theorem 1.1. *Let Ω be a bounded $C^{1,1}$ domain. Suppose that A satisfies the ellipticity condition (1.4) and periodicity condition (1.5). Given $g \in H^1(\Omega)$ and $f \in H^{3/2}(\partial\Omega; \mathbb{R}^d)$ satisfying the compatibility condition (1.2), for $F \in L^2(\Omega; \mathbb{R}^d)$, let $(u_\varepsilon, p_\varepsilon)$, (u_0, p_0) be weak solutions of Dirichlet problems (1.1), (1.6), respectively. Then*

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le C\varepsilon\|u_0\|_{H^2(\Omega)}, \tag{1.7}$$

where the constant C depends only on d, μ, and Ω.