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# Equations of motion for variational electrodynamics

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**Abstract**

We extend the variational problem of Wheeler–Feynman electrodynamics by generalizing the electromagnetic functional to a local space of absolutely continuous trajectories possessing a derivative (velocities) of bounded variation. We show here that the Gateaux derivative of the generalized functional defines *two* partial Lagrangians for variations in our generalized local space, one for each particle. We prove that the critical-point conditions of the generalized variational problem are: (i) the Euler–Lagrange equations must hold Lebesgue-almost-everywhere and (ii) the momentum of each partial Lagrangian and the Legendre transform of each partial Lagrangian must be absolutely continuous functions, generalizing the Weierstrass–Erdmann conditions.


*Keywords:* Calculus of variations; Absolute continuity; Neutral differential-delay equations; State-dependent delay

## 1. Introduction

### 1.1. Significance of the variational formulation

Electrodynamics has *neutral differential-delay equations* of mixed type with implicitly defined *state-dependent* delays for the motion of point charges [1], which theory is still a challenge for present day mathematics. The theory of differential-delay equations with state-dependent delay initiated in the 70's with the foundations based on infinite-dimensional dynamical systems [2–9] and the numerical studies [8,10–12] (see also Ref. [13] for an extensive list of references).

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A formal variational structure for the electromagnetic equations is known since 1903 [1,14], but only recently the variational structure has been embedded into a variational principle [15–17]. The existence of the variational principle is important for the analytic and numerical studies because one is dealing with functional minimization [15,18], which makes the electromagnetic equations special in the class of neutral differential-delay equations.

### 1.2. What is this paper about

Here we invert the direction of application of the variational principle of Refs. [15–17] by extending the local domain of trajectory variations to the set of absolutely continuous orbits possessing velocities of bounded variation. The generalized electromagnetic variational problem is studied for variations belonging to the local normed space $X_{BV}$ of *absolutely continuous* orbits possessing a velocity of *bounded variation*.

The classical problem of the calculus of variations studies a functional of the classical mechanical form, $F \equiv \int_0^T \mathcal{L}(\mathbf{x}, \dot{\mathbf{x}}, t)dt$ [19,20], usually minimized on a domain of continuous and piecewise $C^2$ orbits possessing velocity discontinuities on a finite grid of times (henceforth breaking points). The critical-point conditions of the former functional are (i) at the breaking points the momentum $P \equiv \partial\mathcal{L}/\partial\dot{\mathbf{x}}$ and the Legendre transform $E(\mathbf{x}, \dot{\mathbf{x}}, t)$ of the Lagrangian $\mathcal{L}(\mathbf{x}, \dot{\mathbf{x}}, t)$, defined as $E(\mathbf{x}, \dot{\mathbf{x}}, t) \equiv (\dot{\mathbf{x}} \cdot \partial\mathcal{L}/\partial\dot{\mathbf{x}}) - \mathcal{L}(\mathbf{x}, \dot{\mathbf{x}}, t)$, must be *continuous* functions (henceforth the Weierstrass–Erdmann corner conditions [19,20]) and (ii) the Euler–Lagrange equation should hold on all other points [19,20].

The electromagnetic functional does *not* have the classical mechanical form $F \equiv \int_0^T \mathcal{L}(\mathbf{x}, \dot{\mathbf{x}}, t)dt$ [19], and despite the importance to electrodynamics its critical-point conditions have not been studied in functional analytic detail. While the generalized electromagnetic functional is *not* of the classical mechanical type, we show here that its first variation (the Gateaux derivative) decomposes into a *sum* involving *two* partial Lagrangians of the former type on our extended local space $X_{BV}$, i.e., $\delta S = \delta S_1 + \delta S_2$ with $S_i \equiv \int_0^T \mathcal{L}_i(\mathbf{x}, \dot{\mathbf{x}}, t)dt$ for $i = 1, 2$.

After generalizing the domain of the electromagnetic functional to $X_{BV}$ we prove here that the generalized critical-point conditions are (i) the momentum of each partial Lagrangian and the Legendre transform of each partial Lagrangian must be *absolutely continuous* functions and (ii) the Euler–Lagrange equations must be satisfied Lebesgue-almost-everywhere, which is a well-defined request because velocities of bounded variation have a derivative Lebesgue-almost-everywhere.

References [16,17] studied the variational two-body problem in a domain $X_{\widehat{C}^2}$ of continuous and piecewise $C^2$ orbits possessing discontinuous velocities on a finite grid of times. The critical-point conditions of Refs. [16,17] are Euler–Lagrange equations holding piecewise *and* the Weierstrass–Erdmann corner conditions [19] that the momenta and the partial energies are continuous. The absolute continuity condition is not part of the results of [16,17] and is a stronger version of Weierstrass–Erdmann conditions coming from our extension of the electromagnetic domain to $X_{BV}$.

### 1.3. Existence and uniqueness results

The neutral differential-delay equations with state-dependent delay [1,15] connected to the electromagnetic variational principle are still not well understood in terms of the nature of solutions and their existence and uniqueness. The early studies found a one-parameter family of $C^\infty$