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The existence of solutions to variational problems of slow growth ☆

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Abstract

We consider the existence of solutions, in the space $W^{1,1}(\Omega)$, to the problem

$$\text{minimize} \int_\Omega L(\nabla v(x))dx \quad \text{on} \quad \phi + W_0^{1,1}(\Omega)$$

where L is of slow (linear or at most quadratic) growth. We present a necessary and sufficient condition in order that, for any smooth boundary datum ϕ and for any bounded Ω with smooth boundary, the minimum problem be solvable.

MSC: 49K10; 76M30

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1. Introduction

We consider the existence of solutions, in the space $W^{1,1}(\Omega)$, to the problem

$$\text{minimize} \int_\Omega L(\nabla v(x))dx \quad \text{on} \quad \phi + W_0^{1,1}(\Omega) \tag{1}$$

where L is smooth and of slow (linear or at most quadratic) growth. More precisely, the class $\mathbb{L}$ of Lagrangians L we shall consider is

$$\mathbb{L} = \Big\{ L(\xi) = l(|\xi|) : l : \mathbb{R} \to \mathbb{R}^+ \text{ is strictly convex },\ l(t) = l(-t),$$
$$l \in C^2 \text{ and } l'' \text{ is non-increasing} \Big\}.$$

Lagrangians defined by smooth strictly convex functions l that, for $|t|$ large, grow like $\frac{1}{2}t^2$, or $|t| - \sqrt{|t|} + \gamma$, or $\sqrt{1+t^2}$, all belong to $\mathbb{L}$. The theory of existence of solutions for different Lagrangians, in particular, for some Lagrangians belonging to the class $\mathbb{L}$, is based on different arguments. For Lagrangians of superlinear growth as $L(\xi) = \frac{1}{2}|\xi|^2$, the direct method yields existence of solutions, based on lower semicontinuity and weak compactness, with no mention of the properties either of ϕ or of the boundary of Ω. On the other side of the spectrum, for the non-parametric minimal surface problem, i.e. for $L(\xi) = \sqrt{1+|\xi|^2}$, in order that the minimum problem be solvable for any smooth datum ϕ, a necessary and sufficient condition (see [7]) is that the mean curvature of $\partial\Omega$ be non-positive.

A condition for the existence of solutions (intermediate growth condition) that does not imply superlinear growth was introduced in [4]; the same condition was used in [1] and [2] to prove existence and regularity (lipschitzianity) of solutions to the problem

$$\text{minimize} \int_a^b L(x(t), x'(t))dt;\ x(a) = \alpha,\ x(b) = \beta.$$

The results of the above mentioned papers are based on reparametrizations, an argument specific to one-dimensional integration set and, to these authors' knowledge, this intermediate growth condition has not yet been used for problems on a multi-dimensional integration set Ω.

In this paper we show that this condition (Assumption 1 below) is *necessary and sufficient* in order that, for any smooth boundary datum ϕ and for any bounded Ω with smooth boundary, the minimization problem (1) admits a solution.

In particular, this condition is able to divide Lagrangians of linear growth in two separate classes: those like the non-parametric minimal area problem, $L(\xi) = \sqrt{1+|\xi|^2}$, where the above statement is not true, and those Lagrangians growing, for $|\xi|$ large, like $L(\xi) = |\xi| - \sqrt{|\xi|} + \gamma$, for which we shall prove existence of solutions.

Models with Lagrangians growing linearly are important in elasticity [5]; our u is a scalar and not a vector and we make no claims at solving these problems; still, finding connections might be of some interest.