Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 260 (2016) 6176–6200

Journal of Differential Equations

www.elsevier.com/locate/jde

A differential equation with state-dependent delay from cell population biology

Philipp Getto [a,b,c,∗,1], Marcus Waurick [d]

[a] Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, H-6720 Szeged, Hungary
[b] Center for Dynamics, TU Dresden, 01062 Dresden, Germany
[c] BCAM (Basque Center for Applied Mathematics), 48009 Bilbao, Spain
[d] Department of Mathematical Sciences, University of Bath, BA2 7AY, Bath, UK

Received 29 October 2015; revised 21 December 2015

Available online 6 January 2016

Abstract

We analyze a differential equation, describing the maturation of a stem cell population, with a state-dependent delay, which is implicitly defined via the solution of an ODE. We elaborate smoothness conditions for the model ingredients, in particular vital rates, that guarantee the existence of a local semiflow and allow to specify the linear variational equation. The proofs are based on theoretical results of Hartung et al. combined with implicit function arguments in infinite dimensions. Moreover we elaborate a criterion for global existence for differential equations with state-dependent delay. To prove the result we adapt a theorem by Hale and Lunel to the C^1-topology and use a result on metric spaces from Diekmann et al.

MSC: 34K05; 92C37; 37N25

Keywords: Delay differential equation; State-dependent delay; Semiflow; Global existence; Structured populations; Stem cell model

[∗] Corresponding author at: Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, H-6720 Szeged, Hungary.
E-mail addresses: phgetto@yahoo.com (P. Getto), Marcus.Waurick@tu-dresden.de (M. Waurick).

[1] The research of the author was funded by the DFG (Deutsche Forschungsgemeinschaft) under the project “Delay equations and structured population dynamics”, GE 2382/1-1. He received additional support by the Spanish Ministry of Economy and Competitiveness (MINECO) under project MTM 2010-18318, by the ERC starting grant 658 No. 259559 and by the Fields Institute for Research in Mathematical Sciences under the Short Thematic Program on Delay Differential Equations.

http://dx.doi.org/10.1016/j.jde.2015.12.038
0022-0396/

0. Introduction

In this paper we analyze a class of differential equations of the form

$$w'(t) = q((v(t))w(t), \tag{0.1}$$

$$v'(t) = \frac{\gamma(v(t-\tau(v_t)))g(x_2, v(t))w(t-\tau(v_t))}{g(x_1, v(t-\tau(v_t)))} e^{\int_0^{\tau(v_t)}(d-D_1 g)(y(s,v_t),v(t-s))ds} - \mu v(t). \tag{0.2}$$

We use the standard notation

$$x_t(s) := x(t+s), \;\; s < 0,$$

if a function x is defined in $t+s \in \mathbb{R}$. If t is fixed, then x_t is a function describing the history of x at time t. Both (0.1) and (0.2) are equations in $\mathbb{R}$ and all functions are real-valued. Next, τ is a nonlinear functional with domain in a space of functions. The functions q, γ, g and d have real arguments, μ is a parameter and γ, g, d, τ and μ take nonnegative values.

The functional τ describes the delay and is allowed to depend exactly on the second component v_t of the state. The delay is in general only implicitly given: For a function ψ defined on an interval $[-h, 0]$, we specify $\tau = \tau(\psi)$ as the solution of the equation

$$y(\tau, \psi) = x_1, \tag{0.3}$$

where $y(\cdot, \psi)$ is defined via the ordinary differential equation (ODE)

$$y'(s) = -g(y(s), \psi(-s)), \;\; s > 0,$$
$$y(0) = x_2, \tag{0.4}$$

and $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$ are given model parameters, see Fig. 1. We interpret s as the time to evolve from $y(s)$ to x_2, i.e., we define y going backward in time. This facilitates denoting time dependence in the second argument of g, given that ψ is defined on $[-h, 0]$. As a consequence $y(s, v_t)$ is the state at time $t-s$, given that x_2 is reached at time t. The notation allows to express this state, and hence also the delay τ, as a function of history v_t at time t. Equations (0.1)–(0.4) can be classified as a differential equation with implicitly defined delay with state dependence.

The system describes the maturation process of stem cells. The underlying model is formulated as a partial differential equation (PDE) of transport type in [12]. A special case of the PDE is derived via a limiting argument for related multi-compartment models. In our notation, the PDE formulation is (0.1) along with

$$g(x_1, v(t))u(t, x_1) = \gamma(v(t))w(t),$$
$$\partial_t u(t,x) + \partial_x g(x, v(t))u(t,x) = d(x, v(t))u(x, v(t)), \;\; x \in (x_1, x_2),$$
$$v'(t) = u(t, x_2)g(x_2, v(t)) - \mu v(t).$$

An integration along the characteristics, similar to the one in Section II 4.1 in [24], yields that $u(t, x_2)$ is equal to the first summand on the right hand side of (0.2) divided by $g(x_2, v(t))$. Filling