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Smoothness of semiflows for parabolic partial differential equations with state-dependent delay

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Abstract

In this paper, the smoothness properties of semiflows on C^1-solution submanifold of a parabolic partial differential equations with state-dependent delay are investigated. The problem is formulated as an abstract ordinary retarded functional differential equation of the form $du(t)/dt = Au(t) + F(u_t)$ with a continuously differentiable map G from an open subset U of the space $C^1([-h, 0], L^2(\Omega))$, where A is the infinitesimal generator of a compact C_0-semigroup. The present study is continuation of a previous work [14] that highlights the classical solutions and C^1-smoothness of solution manifold. Here, we further prove the continuous differentiability of the semiflow. We finally verify all hypotheses by a biological example which describes a stage structured diffusive model where the delay, which is the time taken from birth to maturity, is assumed as a function of a immature species population.

Keywords: Partial functional differential equation; State-dependent delay; Manifold; Continuous semiflow; Differentiability

1. Introduction

Delay differential equations play an important role in describing many real-world processes in physical, biological, chemical, control and other problems. It is more realistic to consider the

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past states in addition to the present one in these descriptions. For simplicity, the constancy of delay is usually assumed but this assumption is rarely met in the underlying real-world processes. Therefore assuming state-dependence of time lags is needed, and this was already shown in the example of Antarctic whale and seal populations in [7]. It is observed that for individual of a small seal species it takes three to four years to mature and of large seals it takes five years to mature, of small whales it takes seven to ten years and of large whale species it takes twelve to fifteen years to reach maturity. Besides, Andrewartha and Birch [2] considered how the duration of larval development of flies is viewed as a nonlinear increasing function of larval density. Some basic results on state dependent delay differential equations (SD-DDEs) have been obtained in [10,12,13,16,20,28,29]. In the present work we build the study of smoothness of semiflows for the state-dependent delay partial differential equations (SD-PDEs). Recently, some attention has been paid to SD-PDEs in [14,22–27], which are naturally more difficult to investigate than SD-DDEs. Since the solutions of SD-PDEs usually do not belong to the space of Lipschitz continuous functions, and the time-derivative of a solution belongs to a wider space comparing to the space to which the solution itself belongs, this makes the choice of the appropriate Lipschitz property more involved. In this work, we will combine the results for SD-ODEs [28,29] and SD-PDEs [14] to established our theorems.

In particular, we consider an abstract parabolic partial differential equation with state-dependent time delay. As an example for this class one may consider the equation

$$\begin{cases} \frac{\partial u(t,x)}{dt} = d\Delta u(t,x) + g(u(t-r(u_t),x)), & t>0,\ x\in\Omega, \\ \partial_n u(t,x) = 0, & t>0,\ x\in\partial\Omega \end{cases} \tag{1.1}$$

with $u_0(\cdot, x)$ given. Here $u = col(u_1, \dots, u_n)$, $\Delta u = col(\Delta u_1, \dots, \Delta u_n)$ and $d > 0$ is diffusion coefficient of u; ∂_n is the directional derivative normal to $x \in \partial\Omega$. The homogeneous Neumann boundary condition indicates that there is zero flux across the boundary. It is well known that the second derivative term of (1.1) corresponds to a strongly continuous semigroup of linear operators on a Banach space of functions determined by the boundary conditions.

We will treat the system (1.1) as an abstract ordinary functional differential equation in a Banach space by using the semigroup approaches

$$\begin{cases} \frac{du(t)}{dt} = Au(t) + F(u_t), & t>0, \\ u_0 = u|_{[-h,0]} = \varphi \in C \equiv C([-h,0], L^2(\Omega)), \end{cases} \tag{1.2}$$

with a given delay functional $r : U \to [0, h]$ and the map $F(\varphi) = g(\varphi(-r(\varphi)))$ on a subset U of the Banach space $C([-h, 0], L^2(\Omega))$ of a continuous map $\varphi : [-h, 0] \to L^2(\Omega)$, $h > 0$, and $F : C \to L^2(\Omega)$, with norm

$$\|\varphi\|_c = \max_{\theta\in[-h,0]} \|\varphi(\theta)\|_{L^2}.$$

We use the standard notation $u_t(\theta) = u(t + \theta)$ for $\theta \in [-h, 0]$. The function r expresses the delay depending on the values of the solution u at arguments $s \in [t - h, t]$. If A is a linear operator from $L^2(\Omega)$ to $L^2(\Omega)$, then $\mathcal{D}(A)$ denotes its domain. $B(L^2(\Omega), L^2(\Omega))$ will denote the space of bounded linear everywhere defined operators from $L^2(\Omega)$ to $L^2(\Omega)$ and if $A \in B(L^2(\Omega), L^2(\Omega))$, then $\|A\|$ is the norm of A. We assume that