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Bifurcation points of a singular boundary-value problem on (0, 1)

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Abstract

Following earlier work on some special cases [17,11] and on the analogous problem in higher dimensions [10,20], we make a more thorough investigation of the bifurcation points for a nonlinear boundary value problem of the form

$$-\{A(x)u'(x)\}' = f(\lambda, x, u(x), u'(x)) \text{ for } 0 < x < 1,$$
$$\int_0^1 A(x)u'(x)^2 dx < \infty \text{ and } u(1) = 0,$$

where, for all $\lambda \in \mathbb{R}$ and $x \in (0, 1)$, $f(\lambda, x, 0, 0) = 0$ and

$$\partial_3 f(\lambda, x, 0, 0) = \lambda - V(x) \text{ and } \partial_4 f(\lambda, x, 0, 0) = 0,$$

so that the formal linearization about a trivial solution $u \equiv 0$ is

$$-\{A(x)v'(x)\}' + V(x)v(x) = \lambda v(x) \text{ for } 0 < x < 1.$$

Even when f is a smooth function of all its variables, standard bifurcation theory does not apply to the problem and the results differ from the usual conclusions. This is because we deal with the case where the coefficient A has a critical degeneracy as $x \to 0$ in the sense that

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$$A \in C([0,1]) \text{ with } A(x) > 0 \text{ for } x \in (0,1] \text{ and } \lim_{x\to 0} \frac{A(x)}{x^2} = a > 0.$$

It was observed in [17,11] that if the exponent 2 is replaced by a value less than 2 then classical bifurcation theory can be used to treat the problem. The paper [17] deals with the case $f(\lambda, x, s, t) = \lambda \sin s$ whereas [11] covers the more general form $f(\lambda, x, s, t) = \lambda F(s)$. Here we admit a much broader class of nonlinearities and some new phenomena appear. In particular, we encounter situations where bifurcation does not occur at a simple eigenvalue of the linearization lying below the essential spectrum.

MSC: 34B16; 34B15; 47H30; 34C23

Keywords: Bifurcation; Degenerate elliptic

1. Introduction

This paper concerns the following boundary value problem,

$$-(Au')' + Vu + n(x,u') + g(x,u) = \lambda\{u + h(x,u)\} \text{ for } 0 < x < 1, \tag{1.1}$$

$$u(1) = 0 \quad \text{and} \int_0^1 A(u')^2 dx < \infty, \tag{1.2}$$

for an unknown function u such that $u \in C^1((0,1])$ and Au' is absolutely continuous on the compact subsets of $(0,1]$. It is singular at $x = 0$ because we suppose that

(A) $A \in C([0,1])$ with $A(x) > 0$ for $x > 0$ and $\lim_{x\to 0} \frac{A(x)}{x^2} = a > 0$.

Hence there exist constants $C_2 \geq C_1 > 0$ such that

$$C_1 x^2 \leq A(x) \leq C_2 x^2 \text{ for all } x \in [0,1]. \tag{1.3}$$

As we show in Section 2, (A) and the conditions (1.2) imply that $u \in L^2(0,1)$.

The nonlinear terms n, g and h are of higher order in the sense that

$$\lim_{s\to 0} \frac{n(x,s)}{s} = \lim_{s\to 0} \frac{g(x,s)}{s} = \lim_{s\to 0} \frac{h(x,s)}{s} = 0$$

and they satisfy some additional conditions introduced in Section 3. The potential V is such that

(V) $V \in L^\infty(0,1)$ and there exists $V_0 \in \mathbb{R}$ such that

$$\lim_{z\to 0} \|V - V_0\|_{L^\infty(0,z)} = 0.$$