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Maximum principles, extension problem and inversion for nonlocal one-sided equations ☆

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Abstract

We study one-sided nonlocal equations of the form

$$\int_{x_0}^{\infty} \frac{u(x)-u(x_0)}{(x-x_0)^{1+\alpha}}\,dx = f(x_0),$$

on the real line. Notice that to compute this nonlocal operator of order $0<\alpha<1$ at a point x_0 we need to know the values of $u(x)$ to the right of x_0, that is, for $x \geq x_0$. We show that the operator above corresponds to a fractional power of a one-sided first order derivative. Maximum principles and a characterization with

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an extension problem in the spirit of Caffarelli–Silvestre and Stinga–Torrea are proved. It is also shown that these fractional equations can be solved in the general setting of weighted one-sided spaces. In this regard we present suitable inversion results. Along the way we are able to unify and clarify several notions of fractional derivatives found in the literature.

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1. Introduction

We analyze equations of the form

$$\int_{x_0}^{\infty} \frac{u(x)-u(x_0)}{(x-x_0)^{1+\alpha}}\,dx = f(x_0), \tag{1.1}$$

on $\mathbb{R}$. Expressions like the nonlocal operator above are in general connected with different notions of *fractional* derivatives. If the name "derivative" is reasonable, the object defined in (1.1) should satisfy, in our opinion, some of the fundamental properties of the true derivative. Even more, it would be desirable to see the equation in (1.1) as a certain limit of a classical local differential equation. If that were possible, then the theory of partial differential equations could be applied to the classical equation and then obtain as a consequence some properties for the fractional derivative in (1.1). Finally, one of the important tasks would be to find spaces in which we can solve the equation (1.1). In other words, from the point of view of operator theory, something should be said about the inverse operator $f \to u$. Along this paper all these questions are treated. In this flow of ideas, we establish some maximum principles, see Theorem 1.1 and Corollary 1.2, we show that the fractional derivative defined above is a Dirichlet-to-Neumann operator of a local degenerate PDE equation, see Theorem 1.3, and finally we solve the equation in some Lebesgue spaces related with the one-sided nature of the expression (1.1), see Theorems 1.4 and 1.5.

Obviously one of our primary duties is to locate the operator in a framework for which the name "fractional derivative" has sense. In order to do that in a reasonable way let us make some discussions about equations like (1.1).

The expression $d^n y/dx^n$ was introduced by G.W. Leibniz to denote derivatives of higher integer order. A natural thought has been to extend the definition to non-integer values of n. In September 1695, G.F. Antoine, Marquis de L'Hôpital, wrote a letter to Leibniz asking "What if n be $1/2$?". This letter and Leibniz's answer are considered the starting point of *fractional calculus*, see [20]. Since then a lot of effort has been devoted in order to define and apply fractional derivatives and fractional integrals. It is interesting to notice that different notions of fractional derivatives and integrals have been used in Physics. For example in 1823, N.H. Abel used fractional operations in the formulation of the tautochrone problem, see [20].

The 19th century witnessed a lot of activity in the area. The important contribution of Liouville, together with the names of Riemann and Weyl, are constantly present in the theory of fractional calculus. Along this paper we shall consider the following fractional integral operators