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Norm bound computation for inverses of linear operators in Hilbert spaces

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Abstract

This paper presents a computer-assisted procedure to prove the invertibility of a linear operator which is the sum of an unbounded bijective and a bounded operator in a Hilbert space, and to compute a bound for the norm of its inverse. By using some projection and constructive a priori error estimates, the invertibility condition together with the norm computation is formulated as an inequality based upon a method originally developed by the authors for obtaining existence and enclosure results for nonlinear partial differential equations. Several examples which confirm the actual effectiveness of the procedure are reported.

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1. Introduction

Let X, Y be complex Hilbert spaces endowed with the inner products $(u,v)_X$, $(u,v)_Y$ and the norms $\|u\|_X = \sqrt{(u,u)_X}$, $\|u\|_Y = \sqrt{(u,u)_Y}$, respectively, and let $D(\mathcal{A})$ be a complex Banach space. Assume that $D(\mathcal{A}) \subset X \subset Y$ and that the embedding $D(\mathcal{A}) \hookrightarrow X$ is compact. Let linear operators $\mathcal{A} : D(\mathcal{A}) \to Y$ and $\mathcal{Q} : X \to Y$ be given. This paper will consider a linear operator defined by

$$\mathscr{L} := \mathcal{A} + \mathcal{Q} : \qquad D(\mathcal{A}) \to Y, \tag{1}$$

and propose a procedure for proving invertibility of $\mathscr{L}$, and for computing a constant $M > 0$ satisfying

$$\|\mathscr{L}^{-1}\phi\|_X \leq M\|\phi\|_Y, \qquad \forall\phi \in Y, \tag{2}$$

i.e. a bound for the operator norm of $\mathscr{L}^{-1} : Y \to X$.

In the context of computer-assisted proofs for nonlinear equations, the operator $\mathscr{L}$ stands for the linearization of a given nonlinear problem, and the verification of the invertibility of $\mathscr{L}$ and the computation of a norm bound for $\mathscr{L}^{-1}$ play an essential role in, for example, Newton-type or Newton–Cantorovich-type arguments which aim at proving the existence of a solution of the nonlinear problem with a mathematical rigorous error bound [4,11,15,16,18,19,22].

Our proposed approach is in fact an extension of the methods presented in [10,11,14,27] which are based on finite dimensional spectral norm estimation for Galerkin approximations to $\mathscr{L}^{-1}$. Our bounds are expected to converge, as the Galerkin space increases, to the exact operator norm of $\mathscr{L}^{-1}$ and to provide accurate and efficient enclosure results for the solution of nonlinear problems. The procedure uses numerical means, but all numerical errors are taken into account, and hence it implies rigorous proofs of all statements made. We also note that our proposed method for invertibility and norm bounds has applications to eigenvalue exclosures in Hilbert spaces [28].

Other computational approaches to bounds for $\mathscr{L}^{-1}$ have already been proposed by one of the authors [16,18,19] and Oishi [15,22], for example. The method described in [16,18,19] is based on eigenvalue bounds, which are obtained by the Rayleigh–Ritz and the Lehmann–Goerisch method with additional base functions and some homotopic steps, together with verified computations for rather small matrix eigenvalue problems. It does not need any infinite dimensional projection error estimates and is applicable to differential equation problems on unbounded domains. However when $\mathscr{L}$ is non-self-adjoint, higher-order base functions are needed since then eigenvalue bounds for $\mathscr{L}^*\mathscr{L}$ are required. In contrast, the verified computation for obtaining M used in the present paper does not need higher-order finite dimensional spaces since it is based on the weak formulation.

Oishi's method bounds the operator norm of $\mathscr{L}^{-1}$ by estimations based on numerical computation of the matrix norm of its Galerkin approximation, together with error bounds for the Galerkin projection, similar to the approach used in [8] already. This procedure, in principle, could be applicable to general Banach spaces and operators, and has also connections with the ideas of the present paper.

The paper is organized as follows. Section 2 describes assumptions on the given linear operator and introduces some finite dimensional approximation subspaces. Section 3 is concerned with a criterion to verify the invertibility of $\mathscr{L}$ and to compute norm bounds for $\mathscr{L}^{-1}$ by using