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Classification and stability of relative equilibria for the two-body problem in the hyperbolic space of dimension 2

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Abstract

We classify and analyze the stability of all relative equilibria for the two-body problem in the hyperbolic space of dimension 2 and we formulate our results in terms of the intrinsic Riemannian data of the problem.

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1. Introduction

This paper analyzes the relative equilibria of the generalization of the two-body problem to the complete, simply connected, hyperbolic, two-dimensional space $\mathcal{H}^2$. This is a continuation of the research program initiated in Diacu, Pérez-Chavela, Santoprete [12], on the study of relative equilibria for the n-body problem in spaces of constant curvature. Here we focus on the simplest case of two particles in $\mathcal{H}^2$. We present a rigorous geometric treatment that allows us to fully classify and determine the stability properties of all relative equilibria.

This problem has a long history beginning with the formulation of the Kepler problem in spaces of constant curvature. Apparently (see [10]), the analytic expression of the generalization of Newton's potential for the Kepler problem to the three-dimensional hyperbolic space $\mathcal{H}^3$ was first given by Schering [32], following the geometric ideas suggested decades earlier in the works of Bolyai [2] and Lobachevsky [22]. The dynamics of this problem is integrable and was thoroughly considered by Liebmann [21], and more recently by Kozlov et al. [18] and Cariñena et al. [6]. Just like in the euclidean case, the bounded orbits are (hyperbolic) conics and a version of Kepler's laws holds. A nice explanation for this analogy in terms of projective geometrical arguments can be found in [1]. We also mention that the ideas behind the generalization of the Kepler problem to constant curvature spaces have been taken further into the subriemannian realm [27].

The formulation of the n-body problem in spaces of constant curvature follows by generalizing the Kepler potential to include the pairwise interaction between all the masses [18]. In Section 3 of reference [18], the authors indicate the interest in the investigation of its particular solutions. An explicit expression of the equations of motion can be found in [12]. Physical and mathematical motivations to study this problem, as well as some historical details, can be found in [10].

We remark that apart from the Kepler and the n-body problem, other mechanical systems with configuration manifold a space of constant curvature of dimension two have been discussed very recently, using differential geometric tools. This is the case, for instance, of the two center problem, the harmonic oscillator, and the n vortex problem. See [3,7,8,26,37] and the references therein.

Contrary to the euclidean case, the two-body problem in $\mathcal{H}^2$ does not reduce to the Kepler problem and is non-integrable [34]. A number of publications have addressed the dynamics of the restricted two-body problem in $\mathcal{H}^2$ (see [4,17] and the references therein). Just like the full two-body problem, it is nonintegrable [23].

The impossibility to reduce the two-body problem in $\mathcal{H}^2$ to the Kepler problem has been associated to the absence of "the integral of the center of mass" in [10,11]. From the geometric mechanics perspective, this seems to be related to the algebraic differences between the group of orientation preserving isometries of euclidean two-space and of $\mathcal{H}^2$ (see Remark 4.3). For a treatment of the symmetry reduction of the problem see [5,33] (see also [35] and the references therein).

Previous results on the existence of relative equilibria for the two-body problem in $\mathcal{H}^2$ first appeared in [12] in the case of equal masses and in [14] in the general case. The results in [12] are obtained in the Weierstrass model for $\mathcal{H}^2$ that arises as an embedding of a hyperboloid in Minkowski's 3-dimensional space. The authors work with global coordinates in the ambient Minkowski space and thus refer to them as *extrinsic coordinates*. On the other hand, the treatment in [14] is performed in the Poincaré disk and upper half-plane models. The authors use the terminology *intrinsic coordinates* and they say that they take an *intrinsic approach* since these