Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 260 (2016) 6787–6799

Journal of Differential Equations

www.elsevier.com/locate/jde

Evolutionary problems driven by variational inequalities ☆

Zhenhai Liu [a,*], Shengda Zeng [a], Dumitru Motreanu [b]

[a] *Guangxi Key Laboratory of Universities Optimization Control and Engineering Calculation, and College of Sciences, Guangxi University for Nationalities, Nanning 530006, Guangxi Province, PR China*

[b] *Département de Mathématiques, Université de Perpignan, 66860 Perpignan, France*

Received 19 December 2015

Available online 10 February 2016

Abstract

In this paper we introduce the differential system obtained by mixing an evolution equation and a variational inequality ((EEVI), for short). First, by using KKM theorem and monotonicity arguments, we prove the superpositional measurability and upper semicontinuity for the solution set of a general variational inequality. Then we establish that the solution set of (EEVI) is nonempty and compact. Our approach is based on the theory of semigroups, Filippov implicit function lemma and fixed point theory for set-valued mappings.

Keywords: Evolution equation; Variational inequality; Filippov implicit function lemma; ν-condensing mapping; Hausdorff measure of noncompactness; Set-valued fixed point theory

1. Introduction

In this paper we formulate a new type of problems that consists of an evolution equation driven by a variational inequality ((EEVI), for short)

☆ Project supported by NNSF of China Grant Nos. 11271087, 61263006, NSF of Guangxi Grant No. 2014GXNSFDA118002, the National Science Center of Poland under Maestro Advanced Project No. UMO-2012/06/A/ST1/00262 and Special Funds of Guangxi Distinguished Experts Construction Engineering.

* Corresponding author.

E-mail addresses: zhhliu@hotmail.com (Z. Liu), zengshengda@163.com (S. Zeng), motreanu@univ-perp.fr (D. Motreanu).

http://dx.doi.org/10.1016/j.jde.2016.01.012

0022-0396/

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t), u(t)), & t \in [0, T], \\ u(t) \in S(K, g(t, x(t), \cdot), \phi), & t \in [0, T], \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where $S(K, g(t, x(t), \cdot), \phi)$ stands for the solution set of the variational inequality ((VI), for short): find $u : [0, T] \to K$ such that

$$\langle g(t, x(t), u(t)), v - u(t)\rangle + \phi(v) - \phi(u(t)) \geq 0, \ \forall v \in K. \tag{1.2}$$

Here, for real Banach spaces E and E_1, K is a convex subset of E_1, $A : D(A) \subset E \to E$ is the infinitesimal generator of a C_0-semigroup e^{At} in E, $\phi : E_1 \to (-\infty, +\infty]$ is a convex lower semicontinuous $\not\equiv +\infty$ functional, and $f : [0, T] \times E \times E_1 \to E$ and $g : [0, T] \times E \times K \to E_1^*$ are fixed mappings.

The solutions of problem (EEVI) are understood in the following mild sense.

Definition 1.1. A pair of functions (x, u), with $x \in C([0, T]; E)$ and $u : [0, T] \to K$ measurable, is said to be a mild solution of problem (EEVI) if

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)} f(s, x(s), u(s))ds, \ t \in [0, T],$$

and $u(s) \in S(K, g(s, x(s), \cdot), \phi)$, $s \in [0, T]$. If (x, u) is a mild solution of problem (EEVI), then x is called the mild trajectory and u the variational control trajectory.

Regarding problem (EEVI), we point out that aspects related to it have been until now examined only in a finite dimensional context when $E = \mathbb{R}^n$, $E_1 = \mathbb{R}^m$, $A = 0$ (see Gwinner [6,7], Li, Huang and O'Regan [12], Pang and Stewart [18]). A special mention deserve the results in Liu, Loi and Obukhovskii [14], which are devoted to the case $E = \mathbb{R}^n$, $E_1 = \mathbb{R}^m$, $A = 0$, and $f(t, x(t), u(t)) = g(t, x(t)) + B(t, x(t))u(t)$. Actually, in these works an ordinary differential equation is parameterized by an algebraic variable required to solve a finite-dimensional variational inequality in the state variable of the differential equation. Such problems taking into account simultaneously both dynamics and constraints in the form of inequalities arise in various situations such as electrical circuits with ideal diodes, Coulomb friction problems for contacting bodies, economical dynamics, dynamic traffic networks, control systems. For more details we refer to [1,2,6–9,12–14,17–19].

The purpose of the present paper is to investigate the properties of solution set for the general problem (EEVI). The main point is that (EEVI) is described by a partial differential equation (1.1), with A for example the Laplace operator, subject to an infinite-dimensional variational inequality (1.2). This incorporates large classes of problems and models and here we extend the previous works dealing with such type of problems.

First, we prove essential properties as the superpositional measurability and upper semicontinuity for the solution set of a general variational inequality. Then we establish our main result stating that the solution set of problem (EEVI) is nonempty and compact. In our approach we make use of various analytical and topological tools such as KKM theorem, monotonicity, theory of semigroups, measure of noncompactness, Filippov implicit function lemma and fixed points for set-valued mappings.