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J. Differential Equations 260 (2016) 6855–6871

Journal of Differential Equations

www.elsevier.com/locate/jde

Varieties and analytic normalizations of partially integrable systems

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Received 15 November 2015

Available online 21 January 2016

Abstract

For analytic differential systems $\dot{x} = Ax + f(x)$ in $\mathbb{F}^n$ with $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, we study the varieties of their partial integrability in a neighborhood of the origin. We also prove the existence of analytic normalizations of partially integrable differential systems.

MSC: primary 34A25, 34C20; secondary 34C05, 34C41, 34A34

Keywords: Analytic differential equations; Partial integrability; Variety; Center on a center manifold; Analytic normalization

1. Introduction and statement of the main results

For an analytic differential system

$$\frac{dx}{dt} = \dot{x} = Ax + f(x) = F(x), \qquad x \in \mathbb{F}^n, \tag{1}$$

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http://dx.doi.org/10.1016/j.jde.2016.01.009
0022-0396/

with $A \in M_n(\mathbb{F})$ and $f(x) = o(|x|)$ an n dimensional vector valued analytic function, the problem on its integrability at the origin is classical [1,10,11] and has attracted lots of attention, see e.g. [4,6,8,12,13,15–17] and the references therein. Here $M_n(\mathbb{F})$ represents the set of all $n \times n$ matrices with entries in $\mathbb{F}$. In what follows we denote by $\mathcal{X}$ the vector field associated to system (1).

In the study of locally analytic integrability of analytic differential systems around a singularity, the theory of Poincaré–Dulac normal forms has been playing an important role. Poincaré–Dulac normal form theorem states that *if A is in Jordan normal form, and $f(x)$ is analytic or a formal series, then system (1) can be transformed to its Poincaré–Dulac normal form by a near identity transformation* (*analytic or formal*). Recall that system (1) is in *Poincaré–Dulac normal form* if A is in Jordan normal form and any monomial $x^m e_j$ in the jth component of $f(x)$ is resonant, i.e. $\langle m, \lambda\rangle = \lambda_j$, where λ is the n-tuple of eigenvalues of A, $x^m = x_1^{m_1} \dots x_n^{m_n}$ for $x = (x_1, \dots, x_n)$ and $m = (m_1, \dots, m_n) \in \mathbb{Z}_+^n$ with $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, and $\langle \cdot, \cdot \rangle$ denotes the inner product of two vectors. A near identity transformation is the one of the form $x = y + \varphi(y)$ with $\varphi = o(|y|)$.

The origin of system (1) is a *nondegenerate singularity* if the n-tuple of eigenvalues λ of A are all not zero. In the case when system (1) is a two dimensional one, the origin is *monodromic* if all orbits near it are either ovals or spiral around it. One of classical results of Poincaré and Lyapunov provides an equivalent characterization for the origin of the planar differential system (1) to be a center.

Theorem A. *Assume that the origin of the planar analytic differential system (1) is nondegenerate and monodromic. Then the origin is a center if and only if the system has an analytic first integral in a neighborhood of the origin, and if and only if the system is locally analytically equivalent to its Poincaré–Dulac normal form.*

A similar result for a saddle of planar analytic Hamiltonian systems was obtained by Moser [9]. Some recent generalizations of Theorem A to any finite dimensional analytic differential systems (1) can be found in [4,8,16,17].

When system (1) is a real planar and analytic one with the origin monodromic and nondegenerate, after an invertible linear change of coordinates it can be written in the form

$$\dot{x}_1 = -x_2 + \sum_{k+s=2}^{\infty} \alpha_{ks} x_1^k x_2^s, \quad \dot{x}_2 = x_2 + \sum_{k+s=2}^{\infty} \beta_{ks} x_1^k x_2^s. \tag{2}$$

Straightforward calculations show that there exists an analytic function or a formal series $V(x)$ such that

$$\mathcal{X}(V) = \sum_{j=m}^{\infty} c_j (x_1^2 + x_2^2)^j,$$

with $2 \le m \le \infty$, and $c_j(\alpha_{ks}, \beta_{ks})$ real functions in the coefficients α_{ks} and β_{ks} of system (2). If $m = \infty$ then V is an analytic or a formal first integral of system (2). It was shown by Poincaré and Lyapunov that if for some values of α_{ks} and β_{ks}, all c_j are equal to zero then all trajectories in a neighborhood of the origin are ovals (system (2) has a center at the origin). If there is a $c_j \neq 0$ then all trajectories near the origin spiral (system (2) has a focus at the origin). This geometric